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HIGH SCHOOL MATHEMATICS

Teachers' Edition

UNIT 8

University of Illinois Committee on School Mathematics

Max Beberman, Director

Herbert E. Vaughan, Editor

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UNIT 3

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TEACHERS COMMENTARY

Introduction

A sequence is a function whose domain is the set of positive integers [or, more loosely, a function whose domain is the set of integers greater than some given integer]. Recursive definition and mathematical induction, studied in Unit 7, are techniques for defining sequences and for proving theorems about sequences, respectively. In Unit 8 these techniques are applied to continued sum sequences [section 8.01] and continued product sequences [section 8.02].

In conventional algebra courses, continued sum sequences occur, implicitly, in the work on arithmetic and geometric progressions--an arithmetic progression, for example, is a sequence of a certain kind, and its continued sum sequence is the function s whose value for each positive integer n is the sum, s_n , of its first n terms. One of the simplest examples of a continued product sequence is the factorial sequence--the sequence whose value for each n is the product of the first n positive integers. Another example of a continued product sequence is the sequence s of positive integral powers of a given real number.

In line with the illustrations given in the preceding paragraph, section 8.01 includes the usual material on arithmetic progressions, but placed in proper perspective as an illustration of a more general problem. Section 8.02 includes the usual material on integral exponents. [Nonintegral rational exponents, irrational exponents, and logarithms are dealt with in Unit 9.] Work on geometric progressions falls in section 8.02 because of its dependence on exponentiation. There is, there, a careful development of the theory behind assigning sums to "infinite" geometric progressions. The theorem--Theorem 153--which is basic for finding sums of "finite" geometric progressions suggests methods of factoring differences of like powers, and there follows, consequently, additional work on factoring. The remaining nonoptional material deals with combinatorial problems--including work on "combinations" and "permutations". Students have already had some experience with such problems in Units 5 and 7 [p. 5-42 et seq., and p. 7-71 et seq.] and now have an opportunity to consolidate and extend their knowledge of ways to deal with them. Also, combinatorial problems suggest the binomial theorem--Theorem 178.

As in Unit 7, there is considerable optional material and many sets of Miscellaneous Exercises distributed throughout the text.

The remarks on pedagogical matters in the Introductions to the COMMENTARY for Unit 7 are again relevant. Please re-read TC[7-i] through TC[7-iii]. As was the case in earlier units, the COMMENTARY for Unit 8 contains much that can add to your understanding of the text and of the reasons why the text is written as it is. You may find it helpful to begin by skimming the text, with the help of the Table of Contents, to establish some landmarks, and then start over to study the text and COMMENTARY more carefully. Remember that a worthwhile degree of understanding of a mathematical concept can seldom be acquired [by anyone] at the first reading. Expect that you will need to return to the same topic again and again, each time with the benefit of more experience, before becoming satisfied with your knowledge of it. Familiarize yourself with the optional material--if you don't have class time for it, it may still give you valuable insights.

Here is a tentative schedule for teaching Unit 8 in 40 class sessions [ignoring most optional material, tests, and Review Exercises]. It is offered as a first approximation to help you in organizing your work. The pace this schedule requires is high, and you may well feel the need to devote more time to some portions of the subject. Also, you will certainly want to spend some time on the Review Exercises at the end of the unit. Some of the Miscellaneous Exercises which are spaced throughout the text should be included in your assignments. [You will probably not have time for all of them.] It is also handy to assign these exercises on days when, because class discussion has not proceeded as rapidly as you expected, students are not quite ready to tackle the exercises you planned to assign.

1. Discuss pages 8-1 through 8-7, doing some exercises in class and assigning the remainder, through Exercise G7 on page 8-8. [Part F and, to some extent, Part G are exploration exercises for the recursive definition near the top of page 8-9.]
2. Discuss previous assignment and pages 8-8 through 8-12. Do Part A in class. Assign Parts B, C, D, and F.
3. Do the discovery part of $\star E$ on page 8-13 in class [see TC[8-13]d], and suggest completing the proof part as optional work. Do Part G on page 8-14 in class, and discuss text through page 8-18. Assign Parts A and B.
4. Discuss previous assignment [B2(a) is important discovery work; the later parts of B4, and B7, may have been difficult]. Discuss pages 8-22 and 8-24. Assign Parts C and D on page 8-23 and some of the Miscellaneous Exercises, which begin on page 8-26, including some of Part B.

5. Discuss Part B on page 8-27 as suggested on TC[8-27]e. Do Exploration Exercises on page 8-31 and discuss pages 8-32 and 8-33. Assign Parts A, B, and C on pages 8-34 and 8-35.
6. Discuss pages 8-35 and 8-36, and do most of Parts A through E in class. Discuss pages 8-39 and 8-42. Assign exercises on pages 8-37 and 8-38 which were not done in class, Parts A and B on pages 8-39 through 8-41, and Parts D and E on page 8-43. [Note that E5 and E6 are exploration exercises for text on page 8-43.]
7. Discuss pages 8-43 and 8-44. Do Parts A [orally] and B on page 8-45 in class. Assign Part C on page 8-46 [and Miscellaneous Exercises].
8. Discuss pages 8-52 and 8-53. Do Part A on pages 8-53 and 8-54, orally, in class. Assign Parts B and C. [You may wish to point out the connection between Part ☆D and Part ☆E on page 8-13. Part ☆D has much to offer students in training for discovery, and should be assigned if time permits.]
9. Discuss 8-57, and do Parts A, B, and C in class [mainly oral]. Seat work on Exploration Exercises. Discuss pages 8-60, 8-61, and 8-62. Assign Part A on pages 8-62 and 8-63.
10. Do Parts B, C, and D in class, and discuss page 8-66. Do Parts A, B, and C on pages 8-66 and 8-67 in class. Assign Parts D, E, and F.
11. Do Parts G and H in class. Discuss text on page 8-73, and do Part K in class. Discuss page 8-74 and assign Part L and, for reading, text from middle of page 8-76 to middle of page 8-77.
12. Discuss pages 8-76 through 8-78. Do Exercises A1 and B1 in class, and assign remainder of Parts A and B, and Part C.
13. Discuss pages 8-80 through 8-82. Assign Parts D and F.
14. Discuss previous assignment and do some Miscellaneous Exercises in class. Assign some Miscellaneous Exercises.
15. Do Exploration Exercises in class. Discuss pages 8-92 through 8-94. Do Part A in class. Assign Part B.
16. Discuss Part B. Assign Part C. Do Part D in class.

17. Discuss pages 8-99 through 8-101. Do Part A in class [orally]. Do Exercise B1 in class. Assign B2 and Parts C, D, E, and F.
18. Discuss text on pages 8-102 and 8-103. Assign Exercises in Part G; G1, 2 to one group of students, G3, 4 to another, etc. Do H1, 2 in class. Assign H3-9.
19. Do some of Part A of Miscellaneous Exercises [which begins on page 8-108], in class. Assign Parts B, C, and D.
20. Discuss pages 8-113 and 8-114. Do Part A on page 8-115 orally. Assign Parts B and C.
21. Discuss pages 8-116 through 8-119. Assign exercises of Part A to groups of students. Start Part B in class. Assign some of Part B and all of Part C. [Complete Part B in later assignments.]
22. Discuss pages 8-122 and 8-123, and do Exercises 1-5 and 8 of Part D in class. Assign the remainder of Part D, some of Part B [preceding] and some Miscellaneous Exercises.
23. Do some of Part F, beginning on page 8-124, in class [see COMMENTARY], and assign what is not done.
24. Discuss pages 8-129 and 8-130. Do Exercises 1, 2, and 3 of Part A in class. Assign the remainder of Part A, and Part C.
25. Discuss examples on pages 8-133 and 8-134. Assign Part D. Assign exercises of Part E to different subsets of the class.
26. Discuss Part E and pages 8-138 through 8-141. Assign Parts A, B, and C on page 8-142.
27. Discuss Part C and pages 8-143 and 8-144. Assign Part D and Exercises 1 and 2 of Part E.
28. Discuss Part E and page 8-146. Assign Exercises E3-9 to various subsets, and assign Part F.
29. Discuss Part F and do Miscellaneous Exercises in class. Assign additional Miscellaneous Exercises.
30. Discuss pages 8-158 through 8-162. Do most of Part A in class, orally. Assign remainder of Part A, Part B, some of Part C, and Exercises 1 and 2 of Part D.

1. The first part of the report is a general introduction to the subject of the study. It discusses the importance of the study and the objectives of the research.

2. The second part of the report is a detailed description of the methodology used in the study. It includes information about the sample size, the data collection methods, and the statistical analysis techniques.

3. The third part of the report is a presentation of the results of the study. It includes tables and graphs showing the data and the findings of the research.

4. The fourth part of the report is a discussion of the results and their implications. It discusses the strengths and weaknesses of the study and the potential for future research.

5. The fifth part of the report is a conclusion and a summary of the findings. It provides a final statement on the results of the study and the overall conclusions.

6. The sixth part of the report is a list of references. It includes all the sources of information used in the study, such as books, articles, and websites.

7. The seventh part of the report is an appendix. It contains additional information that is not included in the main body of the report, such as raw data or detailed calculations.

8. The eighth part of the report is a glossary. It defines the key terms and concepts used in the study, ensuring that the reader understands the terminology.

9. The ninth part of the report is a list of figures. It includes all the graphs and charts used in the study, providing a visual representation of the data.

10. The tenth part of the report is a list of tables. It includes all the tables used in the study, providing a structured way to present the data.

11. The eleventh part of the report is a list of abbreviations. It defines the abbreviations used throughout the report, making it easier for the reader to understand the text.

12. The twelfth part of the report is a list of acknowledgments. It thanks the individuals and organizations that provided support and assistance during the study.

13. The thirteenth part of the report is a list of appendices. It includes all the additional information that is not included in the main body of the report, such as raw data or detailed calculations.

14. The fourteenth part of the report is a list of references. It includes all the sources of information used in the study, such as books, articles, and websites.

31. Do remainder of Part C in class, discuss Part D. Do some of Part E. Assign remainder of Part E, and Parts F and G.
32. Discuss pages 8-166 through 8-168. Do some of each of Parts A, B, C, and D in class. Assign the remainder, together with Part F. Also, assign some Miscellaneous Exercises.
33. Discuss text from bottom of page 8-169 through page 8-172. Do Parts G and H in class. Discuss pages 8-173 and 8-174. Assign Exercises 1-6 of Part A.
34. Discuss previous assignment. Do Exercises 7-12 of Part A in class. Discuss page 8-177. Assign Part B.
35. Discuss pages 8-179 and 8-180. Assign Part C.
36. Discuss pages 8-182 and 8-183. Assign Part D.
37. Discuss pages 8-184 through 8-186. Assign Exercises on page 8-186.
38. Discuss pages 8-196 and 8-197. Assign Parts A and B, and some of Part C.
39. Discuss previous assignment. Do Part F in class. Assign remainder of Part C and Parts D and E.
- ★40. Discuss pages 8-207 through 8-210.

The foregoing should take from 10 to 12 weeks. It is likely that the sequence of lessons described above will need to be interrupted with lessons given over to consolidation and individual help.

1. The first part of the report is a general introduction to the subject of the study.

2. The second part of the report is a detailed description of the methods used in the study.

3. The third part of the report is a discussion of the results of the study.

4. The fourth part of the report is a conclusion and a list of references.

5. The fifth part of the report is a list of appendices.

6. The sixth part of the report is a list of figures.

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8. The eighth part of the report is a list of footnotes.

9. The ninth part of the report is a list of acknowledgments.

10. The tenth part of the report is a list of references.

11. The eleventh part of the report is a list of appendices.

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qth positive odd number is $2q - 1$.] First, we prove that the "sum" of the first 1 positive odd number(s) is 1^2 . Since $1^2 = 1$, we have it. Now, supposing that the sum of the first q positive odd numbers is q^2 , we wish to show that the sum of the first q + 1 positive odd numbers is $(q + 1)^2$. Since the qth positive odd number is $2q - 1$, the (q + 1)th positive odd number is $2(q + 1) - 1$, or $2q + 1$. So, the sum of the first q + 1 positive odd numbers is $q^2 + (2q + 1)$, or $(q + 1)^2$. Hence, if the sum of the first q of them is q^2 then the sum of the first q + 1 of them is $(q + 1)^2$. So, by the PMI, the generalization in question follows.

In the second case, the relevant generalization is that, for each n, the sum of the first n terms of the sequence is $n/(n + 1)$. The case for $n = 1$ is obvious. Now, suppose that the sum of the first q terms is $q/(q + 1)$. Since the qth term is $1/[q \cdot (q + 1)]$, the (q + 1)th term is $1/[(q + 1)(q + 2)]$. So, the sum of the first q + 1 terms is $q/(q + 1) + 1/[(q + 1)(q + 2)]$. That is [by algebra], it is $(q + 1)/(q + 2)$. Hence, if the sum of the first q terms is $q/(q + 1)$ then the sum of the first q + 1 terms is $(q + 1)/(q + 2)$. So, by the PMI, the generalization follows.

The proofs of these generalizations are handled with much more finesse later in the unit, thanks to the introduction of Σ -notation, but a preliminary informal treatment like that described above helps prepare the way.

The sum of the first 60 positive odd numbers is 60^2 and the sum of the first 90 is 90^2 . Use as many examples as necessary to get students to discover the generalization. Giving examples out of sequence is a useful device for determining if students have the generalization without actually asking them to verbalize it. For example, a student might give very quickly the answer '36' for the sum of the first 6 positive odd numbers, and his answer might have been obtained just by adding 11 to 25. So, the next thing to ask him for is the sum of the first 15 positive odd numbers or the sum of the first 60 positive odd numbers.

The sums in order for the second sequence are $1/2$, $2/3$, $3/4$, and $4/5$. The sum of the first 5 terms is $5/6$, that for the first 6 terms is $6/7$, that for the first 1000 terms is $1000/1001$, and the sum for the first 1000000 terms is $1000000/1000001$. We guess that, for each n , the sum of the first n terms is $n/(n + 1)$. Since, for each n , $0 < n < n + 1$, $n/(n + 1) < 1$. So, we guess that no matter how far out in the sequence one goes, he cannot get a sum greater than 1.

*

Both of these introductory problems are handled in more formal fashion later in the unit--the second on page 8-8, and the first in Part A on page 8-12. Although this introduction is largely motivational and attempts to lead students into the work on continued sums [section 8.01] after defining sequences and giving practice in recursive definitions, your students may want to attack immediately the problem of proving the generalizations they discovered. If so, after mentioning that this will be done again later, go to it. Clearly, these are proofs which require mathematical induction.

In the first case, we want to prove that, for each n , the sum of the first n positive odd numbers is n^2 . [In part (ii) of the proof you will need to talk about "the q th positive odd number". Before becoming involved in the proof, have students suggest formulas and reach agreement that the

line 10: $O_4 = 2 \cdot 4 - 1 = 7$; $O_{96} = 191$; ' O_{-3} ' is not defined since -3 is not in the domain of the sequence O . [See page 7-62 of Unit 7 for a discussion of the type of functional notation exemplified by ' O_4 '.]

line 12b: Note that the meaning of 'values' in this sentence is different from its meaning in lines 2 and 3. The values of a variable are the things whose names may be used as replacements for the variable [the values of ' x ' are the real numbers, those of ' n ' are the positive integers, those of ' a ' and ' b ' (in this context) are sequences]. On the other hand, the values of a function are the members of its range--that is, they are the things which are second components of the ordered pairs which belong to the function. [Historically, it seems likely that the use of 'value' in these two senses arose from confusing variables and the functions we call 'variable quantities'.]

line 6b:
$$\begin{cases} a_1 = \frac{1}{2} \\ \forall_n a_{n+1} = a_n \cdot \frac{n}{n+2} \end{cases}$$

*

Answers for Exercises.

1. $8 \cdot 7 \cdot 6$ [or: 336]; 504; 720

2. $\frac{n}{2n+3}$ [Since $b_n = \frac{n-1}{2n+1}$, $b_{\overbrace{n+1}} = \frac{\overbrace{(n+1)} - 1}{2(\overbrace{n+1}) + 1} = \frac{n}{2n+3}$.]; $\frac{n+1}{2n+5}$

[Recall from Unit 7 that the domain of ' m ', ' n ', ' p ', and ' q ' is I^+ and that of ' i ', ' j ', and ' k ' is I . See Exercise 8 on page 8-3.]

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such that, for each x , $f(x) = 4x^2 - 36x + 71$ is symmetric with respect to the line $\{(x, y): x = 9/2\}$ (page 5-177 of Unit 5. A good follow-up problem is provided by changing ' $4n^2 - 36n + 71$ ' to ' $4n^2 - 40n + 71$ '. In this case there is no integer m such that $s_{2m} = 2s_m$. In general, there is such an integer m if and only if the axis of symmetry is $\{(x, y): x = (2p + 1)/2\}$, for some positive integer p .]

14. 1500 [The sequence is 1, 2, 1, 2, 1, 2, ...--that is, each odd-numbered term is 1 and each even-numbered term is 2. Among the first 1000 terms, there are 500 of each kind. So, the sum of the first 1000 terms is $500(1 + 2)$. Ask for the sum of the first 999 terms (1498); the sum of the first 1001 terms (1501).]
15. 0 [The terms are alternately 1 and -1--1, -1, 1, -1, So, the sum of the first 100 terms is $50(1 + -1)$. Ask for the sum of the first 99 terms, of the first 869 terms, of the first 2846 terms. (Answers: 1, 1, 0)]
16. 100^2 ; $200^2 - 100^2$ [Computing hint: $200^2 - 100^2 = 100 \cdot 300$]

9. 3 $[a_{2n} = 3(2n) - 4, a_{2n+1} = 3(2n + 1) - 4; \text{ difference is } 3]$
 If you ask students for a recursive definition of the sequence a , make clear that one needs to compute $a_{n+1} - a_n$ which might (but in this case, doesn't) differ from $a_{2n+1} - a_{2n}$.]

10. 10 [The sequence b is a constant; more explicitly, a constant sequence--that is, a constant function whose domain is I^+ . (See page 5-104 of Unit 5.)]

11. the 1st term; there is no smallest term [Students should see that each term after the first is 10 less than its predecessor. So, the first term, -7 , is the largest, and there is no smallest.]

12. there is no smallest term; the 5th term; three [Since, for each m , $(m - 5)^2 \geq 0$, each term of a is at most 3, and the 5th term is 3. To find how many terms are positive one must find the number of positive integers m such that $3 - (m - 5)^2 > 0$. Equivalently, find the number of positive integers m such that $(m - 5)^2 < 3$ --that is, by Theorem 98, such that $-\sqrt{3} < m - 5 < \sqrt{3}$. Since $-\sqrt{3} < m - 5 < \sqrt{3}$ if and only if $5 - \sqrt{3} < m < 5 + \sqrt{3}$, there are just three such positive integers--4, 5, and 6. So, the positive terms of a are the 4th, 5th, and 6th. Ask students to graph the sequence a .]

★13. 4 [Students should compute the first few terms of the sequence a .

39, 15, -1 , -9 , -9 , -1 , 15, 39, 71

Noting the symmetry, it is easy to see that the sum of the first 8 terms is twice the sum of the first 4 terms. In discussing this exercise, recall to students that the quadratic function f

The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_n = \frac{1}{n!}$. It is shown that $f(x)$ is an entire function and that $f(x) = e^x$. The second part of the paper is devoted to the study of the properties of the function $g(x)$ defined by the equation $g(x) = \sum_{n=0}^{\infty} b_n x^n$, where $b_n = \frac{1}{n!}$. It is shown that $g(x)$ is an entire function and that $g(x) = e^x$.

The third part of the paper is devoted to the study of the properties of the function $h(x)$ defined by the equation $h(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_n = \frac{1}{n!}$. It is shown that $h(x)$ is an entire function and that $h(x) = e^x$. The fourth part of the paper is devoted to the study of the properties of the function $k(x)$ defined by the equation $k(x) = \sum_{n=0}^{\infty} d_n x^n$, where $d_n = \frac{1}{n!}$. It is shown that $k(x)$ is an entire function and that $k(x) = e^x$.

The fifth part of the paper is devoted to the study of the properties of the function $l(x)$ defined by the equation $l(x) = \sum_{n=0}^{\infty} e_n x^n$, where $e_n = \frac{1}{n!}$. It is shown that $l(x)$ is an entire function and that $l(x) = e^x$. The sixth part of the paper is devoted to the study of the properties of the function $m(x)$ defined by the equation $m(x) = \sum_{n=0}^{\infty} f_n x^n$, where $f_n = \frac{1}{n!}$. It is shown that $m(x)$ is an entire function and that $m(x) = e^x$.

The seventh part of the paper is devoted to the study of the properties of the function $n(x)$ defined by the equation $n(x) = \sum_{n=0}^{\infty} g_n x^n$, where $g_n = \frac{1}{n!}$. It is shown that $n(x)$ is an entire function and that $n(x) = e^x$. The eighth part of the paper is devoted to the study of the properties of the function $o(x)$ defined by the equation $o(x) = \sum_{n=0}^{\infty} h_n x^n$, where $h_n = \frac{1}{n!}$. It is shown that $o(x)$ is an entire function and that $o(x) = e^x$.

The ninth part of the paper is devoted to the study of the properties of the function $p(x)$ defined by the equation $p(x) = \sum_{n=0}^{\infty} i_n x^n$, where $i_n = \frac{1}{n!}$. It is shown that $p(x)$ is an entire function and that $p(x) = e^x$. The tenth part of the paper is devoted to the study of the properties of the function $q(x)$ defined by the equation $q(x) = \sum_{n=0}^{\infty} j_n x^n$, where $j_n = \frac{1}{n!}$. It is shown that $q(x)$ is an entire function and that $q(x) = e^x$.

The eleventh part of the paper is devoted to the study of the properties of the function $r(x)$ defined by the equation $r(x) = \sum_{n=0}^{\infty} k_n x^n$, where $k_n = \frac{1}{n!}$. It is shown that $r(x)$ is an entire function and that $r(x) = e^x$. The twelfth part of the paper is devoted to the study of the properties of the function $s(x)$ defined by the equation $s(x) = \sum_{n=0}^{\infty} l_n x^n$, where $l_n = \frac{1}{n!}$. It is shown that $s(x)$ is an entire function and that $s(x) = e^x$.

3. $2m; 6m - 1$ [or: $2(3m) - 1$]

4. the 10th [Since the positive integral root of ' $p^2 - 3p + 5 = 75$ ' is 10, 75 is the 10th term of the sequence b. Ask students which term of b is 16 (Ans: 16 is not a term of b, since ' $p^2 - 3p + 5 = 16$ ' has no positive integral root.) Ask which term of b is 3. (Ans: 3 is both the 1st and the 2nd term of b.)]

5. 60^2 [We hope students will discover that the sequence s is the sequence of sums of the odd positive integers, starting with the first. Thus, $s_{60} = 60^2$, or 3600. If they don't make this discovery, Exercise 5 will take a lot of work.]

6. $1/61$ [We expect students to handle this problem in the same way the problems on page 8-1 are handled. That is, find the first few terms and look for a pattern.

$$1 \text{ term} \rightarrow \frac{1}{2} = \frac{1}{2}$$

$$2 \text{ terms} \rightarrow \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

$$3 \text{ terms} \rightarrow \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{4}$$

So, an inspired guess is that the 60th term is $1/61$.
[Although formal work on continued products does not commence until much later in the unit (section 8.02), an informal proof at this point (like those in the COMMENTARY for page 8-1) would not be amiss.]]

7. $(n + 2)(n + 3)(n + 4)$ [or: $(n + 1 + 1)(n + 1 + 2)(n + 1 + 3)$];
 $(n + 3)(n + 4)(n + 5)$ [or: $(n + 2 + 1)(n + 2 + 2)(n + 2 + 3)$]

8. $\begin{cases} a_1 = 4 \\ \forall n \ a_{n+1} = a_n + 5 \end{cases}$ $\begin{cases} [a_1 = 5 \cdot 1 - 1 = 4, \ a_2 = 5 \cdot 2 - 1 = 5(1 + 1) - 1 = \\ (5 \cdot 1 - 1) + 5 = a_1 + 5, \text{ and, in general,} \\ a_{p+1} = 5(p + 1) - 1 = (5p - 1) + 5 = a_p + 5.] \end{cases}$

It is sometimes helpful in getting students to understand the role of the 'for all $p \leq n$ ' to analyze an instance of the generalization in line 10:

\forall_n the sum of the numbers O_p , for all $p \leq n$, is n^2

Take the "fifth" instance:

the sum of the numbers O_p , for all $p \leq 5$, is 5^2

The portion of this instance up to the word 'is' can be translated into:

$$O_1 + O_2 + O_3 + O_4 + O_5$$

Similarly, the portion up to the word 'is' of the fifth instance of the generalization in line 12 can be translated into:

$$(2 \cdot 1 - 1) + (2 \cdot 2 - 1) + (2 \cdot 3 - 1) + (2 \cdot 4 - 1) + (2 \cdot 5 - 1)$$

*

Note the parentheses in (2).

$$\begin{array}{l} \text{compare} \left\{ \begin{array}{l} \sum_{p=1}^3 (2p - 1) = (2 \cdot 1 - 1) + (2 \cdot 2 - 1) + (2 \cdot 3 - 1) \\ \sum_{p=1}^3 2p - 1 = [2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3] - 1 \end{array} \right. \end{array}$$

Because of our convention to carry out multiplications before additions,

' $\sum_{p=1}^3 2p$ ' is an acceptable abbreviation for ' $\sum_{p=1}^3 (2p)$ '. Also, because of

our conventions to do additions and subtractions from left to right,

' $\sum_{p=1}^3 2p - 1$ ' is an acceptable abbreviation for ' $\left(\sum_{p=1}^3 2p \right) - 1$ '.

and:

$$\forall_p p \geq 1 \iff \forall_m m \geq 1$$

In contrast, the "statement":

$$\sum_{x=1}^3 (2 - 3x) = \sum_{m=1}^3 (2 - 3m)$$

is nonsense. Since the domain of 'x' is the set of real numbers, the expression on the left is nonsense. [$\forall_x x \geq 1$ is, of course, not nonsense, but it is not equivalent to $\forall_m m \geq 1$.] Similarly,

$$\{n: n > 5\} = \{p: p > 5\} \neq \{x: x > 5\} = \{y: y > 5\}$$

*

$$4. \frac{1(1-1)}{2} + \frac{2(2-1)}{2} + \frac{3(3-1)}{2} + \frac{4(4-1)}{2} \quad [\text{or: } \frac{1 \cdot 0}{2} + \frac{2 \cdot 1}{2} + \frac{3 \cdot 2}{2} + \frac{4 \cdot 3}{2}]$$

$$5. \frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \frac{1}{3(3+1)} + \frac{1}{4(4+1)} + \frac{1}{5(5+1)}$$

$$[\text{or: } \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6}]$$

$$6. \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) [\text{or: } \frac{1}{1} - \frac{1}{6}, \text{ or: } \frac{5}{6}]$$

[Here it is instructive to ask for a simplified answer. In fact, you can build a set of discovery exercises by using numbers like 6, 7, 9, 17, and 1000 as upper limits. Exercise 2 of Part D can be anticipated in this way.]

Answers for the rest of Part C are in the COMMENTARY for page 8-6.

1. Introduction

2. Methodology

3. Results and Discussion

4. Conclusion

5. Acknowledgments

6. References

7. Appendix

8. Author Biographies

9. Contact Information

10. Declaration of Interest

11. Supplementary Materials

12. Additional Information

Answers for Part A.

$$1. \quad \forall_n \sum_{p=1}^n (2p + 1) = n^2 + 2n$$

$$2. \quad \sum_{q=1}^2 \frac{1}{7q + 3} = \frac{27}{170}$$

$$3. \quad \sum_{m=1}^5 \frac{1}{m^2 + 1} \neq 81$$

$$4. \quad \sum_{q=1}^2 \frac{1}{q} = 1.5$$

$$5. \quad \sum_{p=1}^1 \frac{1}{p^2} = 1$$

*

Answers for Part B.

1. Sigma, from $p = 1$ to 7, of $3p - 4$ is greater than or equal to 0.

2. Sigma, from $p = 1$ to 8, of p is not sigma, from $p = 1$ to 9, of p .
[Ask students if they think these two sentences (and the ones in Part A) are true.]

*

Answers for Part C.

[Here we want students to practice translating from Σ -notation to indicated-sum-notation. Hence, "unsimplified" answers are acceptable. If we wanted simplifications, the exercise instructions would have been 'Compute. '.]

$$1. \quad 1 + 2 + 3 + 4 + 5$$

$$2. \quad (2 - 3 \cdot 1) + (2 - 3 \cdot 2) + (2 - 3 \cdot 3)$$

3. [same as the answer for Exercise 2]

*

In connection with Exercises 2 and 3, note that in writing Σ -expressions one may use any apparent variable whose domain is I^+ , and that two expressions which differ only in the apparent variables used have the same meaning. Compare true statements such as:

$$\sum_{p=1}^3 (2 - 3p) = \sum_{m=1}^3 (2 - 3m)$$

7. $4 \cdot 1 + 4 \cdot 2 + 4 \cdot 3 + 4 \cdot 4 + 4 \cdot 5 + 4 \cdot 6 + 4 \cdot 7 + 4 \cdot 8 + 4 \cdot 9 + 4 \cdot 10$
 [We trust that some students will factor out the '4' and thus anticipate Theorem 133.]
8. $(3 \cdot 1 + 4) + (3 \cdot 2 + 4) + (3 \cdot 3 + 4) + (3 \cdot 4 + 4)$ [or: $3(1 + 2 + 3 + 4) + 4 \cdot 4$
 This second answer anticipates Theorems 134, 133, and 132a,]
9. $6 \cdot 4$ [or: 24]
 [Note that Exercise 8 paves the way for Exercise 9 which, in turn, prepares students for Exercise 3 of Part D.]

*

Each of the exercises of Part D is intended to suggest a generalization which will be handled formally later in the unit. At this time students should be encouraged to state the generalization suggested by the exercise after they have carried out the verification asked for.

In the case of Exercise 1, the generalization is:

$$\forall_n \sum_{p=1}^n p = \frac{n(n+1)}{2}$$

Students will probably recall this from their work with triangular numbers in Unit 7.

The generalization for Exercise 2:

$$\forall_n \sum_{q=1}^n \left(\frac{1}{q} - \frac{1}{q+1} \right) = \frac{n}{n+1}$$

should have been anticipated in Exercise 6 of Part C on page 8-5.

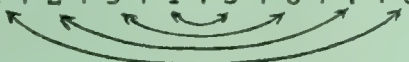
Similarly, the generalization for Exercise 3:

$$\forall_x \forall_n \sum_{p=1}^n x = nx$$

was foreshadowed in Exercise 9 of Part C, and will be called for again in Exercise 5 of Part E. Note that the groundwork for continued sums of constant sequences was laid in Exercise 10 on page 8-3.

*

Answers for Part D.

$$1. \quad 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 4 \cdot 9 = \frac{8 \cdot 9}{2}$$


$$2. \quad \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{10} - \frac{1}{11}\right) = 1 - \frac{1}{11} = \frac{10}{11}$$


3. $4 + 4 + 4 + 4 + 4 + 4 = 6 \cdot 4 = 24$ [As does Exercise 10 on page 8-3, this exercise deals with a constant sequence--the sequence whose value for each (positive integral) argument is 4. Compare with Exercise 5 of Part E.]

*

Answers for Part E.

[The exercises of this part do not have unique answers. For example, as noted in the COMMENTARY for page 8-5, equally correct answers for Sample 1 are:

$$\sum_{q=1}^4 (2 + 3q)$$

$$\sum_{m=1}^4 (2 + 3m)$$

$$\sum_{n=1}^4 (2 + 3n)$$

(But $\sum_{x=1}^4 (2 + 3x)$ is not correct.)]

$$1. \quad \sum_{p=1}^4 (5p + 2)$$

$$2. \quad \sum_{p=1}^5 (1 - 3p^2)$$

$$3. \quad \sum_{p=1}^6 2p$$

$$4. \sum_{p=1}^5 p^3$$

$$5. \sum_{p=1}^6 4$$

$$6. \sum_{p=1}^3 (p^2 - p)$$

$$7. \sum_{p=1}^4 \frac{1}{p(p+1)}$$

$$8. \sum_{p=1}^4 \frac{p}{(p+1)(p+2)}$$

$$9. \sum_{p=1}^5 \frac{p^2}{2p(2p+1)}$$

$$10. \sum_{p=1}^6 p(p+1)(p+2)$$

$$11. \sum_{p=1}^{40} (2p - 1)$$

$$12. \sum_{p=1}^{11} (3p + 2)$$

[Exercises 11-13 exemplify a commonly-found procedure for indicating a continued sum.]

Answers for the rest of Part E are in the COMMENTARY for page 8-7.

$\frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{4}$

$$\frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{4}$$

$$\frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{4}$$

The probability of getting a head in a single trial is $\frac{1}{2}$.

The probability of getting a head in two trials is $\frac{1}{4}$.

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$\frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{4}$

$$\frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{4}$$

$$\frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{4}$$

The probability of getting a head in a single trial is $\frac{1}{2}$.

The probability of getting a head in two trials is $\frac{1}{4}$.

•

Answers for Part F.

[These exercises go further in anticipating the definition of Σ -notation given on page 8-9. Exercises 1 and 2 are designed especially for the first part of that definition. Actually, it is impossible for students to verify the sentences given in Exercises 1 and 2, just as it was impossible for them to say 'true' or 'false' to the sentence in Exercise 5 of Part A

on page 8-5. All that they can say is that if ' $\sum_{q=1}^1 a_q$ ' means the first

term of sequence a , then the sentences in Exercises 1 and 2 are true. This point should be made now in this informal setting to build an awareness of the need for definition, and thus help students appreciate the definition on page 8-9 and the discussion about extension of meaning on pages 8-32 ff. Of course, the sentences in Exercises 3 and 4 can be verified by actual computation, but we hope that the large upper limit in Exercise 3 will discourage this.]

*

The exercises in Part G lay the groundwork for Exercises 3[☆](k), 4, and 7 on pages 8-20, 8-21, and 8-22. They also help to develop facility in the use of Σ -notation. The solutions depend on the fact that, for each positive integer n and each positive integer q , there are exactly q integers m such that $n < m \leq n + q$ [compare with Theorem 112]. Students should become aware of the relevance of this as they do the exercises.

*

Answers for Part G.

1. 7 [The given sentence is equivalent to:

$$\sum_{p=1}^6 p < m \leq \sum_{p=1}^6 p + 7$$

Since $\sum_{p=1}^6 p$ is an integer--say, n --the least integer m which satisfies this sentence is $n + 1$, the next is $n + 2$, ... and the greatest is $n + 7$.]

2. 33

3. 58

4. 255 [255 = $16^2 - 1$]

Answers for the rest of Part G are in the COMMENTARY for page 8-8.

TC[8-7]b

141

142

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$$13. \sum_{q=1}^{10} 2q^3$$

$$14. \sum_{p=1}^6 (p^3 + 1)$$

$$15. \sum_{p=1}^7 [(p-1)^3 + 1]$$

$$16. \sum_{p=1}^5 (2q)^3$$

*

Some students may suggest ' $\sum_{p=0}^6 (p^3 + 1)$ ' as the answer to Exercise 15.

Note that such an answer anticipates the extension of meaning of Σ -notation which is taken up on pages 8-32 ff. With such an extension the variable 'p' is not appropriate in view of the fact that the domain of 'p' is the set of positive integers.

*

Exercises 17-21 anticipate the definition of Σ -notation given on page 3-9. Note that these exercises also force students to pay attention to the scope of the Σ -sign. The scope would be unambiguously indicated by the use of a pair of grouping symbols associated with it. For example, the expression in Exercise 17 is an abbreviation for:

$$\left(\sum_{p=1}^7 (3p + 1) \right) + (3 \cdot 8 + 1)$$

*

$$17. \sum_{p=1}^8 (3p + 1)$$

$$18. \sum_{p=1}^6 (p^2 + 1)$$

$$19. \sum_{p=1}^7 (4p + 1)$$

$$20. \sum_{p=1}^{n+1} p(p + 1)$$

$$21. \sum_{p=1}^{n+4} (p-1)(p+1)$$

*

5. 0 [For any integer n , there are no real numbers--let alone integers-- x such that $n < x \leq n + 1$. We set things right in the next exercise, and prepare students for the description of the sequence a in Exercise 7.]

6. 18

7. a_{99}

*

The explicit definition for the sequence s asked for in the third line from the bottom of page 8-8 is, of course, the generalization which students probably discovered on page 8-1:

$$\forall n \quad s_n = \frac{n}{n+1}$$

The connection between a given sequence a and its continued sum sequence can be illustrated schematically as follows:

$$\begin{array}{c} a_1 + a_2 + a_3 + a_4 + a_5 + \dots + a_n + a_{n+1} + \dots \\ \underbrace{\hspace{1cm}}_{s_1} \\ \underbrace{\hspace{1.5cm}}_{s_2} \\ \underbrace{\hspace{2cm}}_{s_3} \\ \underbrace{\hspace{2.5cm}}_{s_4} \\ \underbrace{\hspace{3cm}}_{s_5} \\ \underbrace{\hspace{4cm}}_{s_n} \\ \underbrace{\hspace{5cm}}_{s_{n+1}} \end{array}$$

*

A pair of sequences, the second of which is the sequence of continued sums of the first, is called a series. The terms of the first sequence are called the terms of the series and the terms of the second sequence are called the partial sums of the series. So, for the sequences b and s of page 8-8, the pair (b, s) is a series whose p th term is b_p and whose n th partial sum is s_n . [We do not make use of this terminology in this unit.]

We give below both column proofs and paragraph proofs for Part A and for Exercise 1 of Part B. Since column proofs for all exercises of this kind are similar [and the same holds for paragraph proofs], we give only the "algebra" parts for Exercises 2, 3, and 4 of Part B. Students should give some column proofs, since doing so will prepare them for the discussion of Theorem 130 which starts on page 8-14. They should also give some paragraph proofs to practice their skill in writing paragraphs.

*

Answer for Part A.

Part (i):

$$(1) \quad \sum_{p=1}^1 (2p-1) = 2 \cdot 1 - 1 \quad [\text{recursive definition}]$$

$$(2) \quad 1^2 = 2 \cdot 1 - 1 \quad [\text{theorem}]$$

$$(3) \quad \sum_{p=1}^1 (2p-1) = 1^2 \quad [(1), (2)]$$

Part (ii):

$$(4) \quad \sum_{p=1}^q (2p-1) = q^2 \quad [\text{inductive hypothesis}]^*$$

$$(5) \quad \forall_n \sum_{p=1}^{n+1} (2p-1) = \sum_{p=1}^n (2p-1) + [2(n+1)-1] \quad [\text{recursive definition}]$$

$$(6) \quad \sum_{p=1}^{q+1} (2p-1) = \sum_{p=1}^q (2p-1) + [2(q+1)-1] \quad [(5)]$$

$$(7) \quad = q^2 + [2(q+1)-1] \quad [(4), (6)]$$

$$(8) \quad = q^2 + 2q + 1 \quad \left. \vphantom{\begin{matrix} (8) \\ (9) \end{matrix}} \right\} \quad [\text{algebra}]$$

$$(9) \quad = (q+1)^2$$

L. H. CRANE, JR., Editor

1. 2. 3. 4. 5. 6. 7. 8. 9. 10. 11. 12. 13. 14. 15. 16. 17. 18. 19. 20. 21. 22. 23. 24. 25. 26. 27. 28. 29. 30. 31. 32. 33. 34. 35. 36. 37. 38. 39. 40. 41. 42. 43. 44. 45. 46. 47. 48. 49. 50. 51. 52. 53. 54. 55. 56. 57. 58. 59. 60. 61. 62. 63. 64. 65. 66. 67. 68. 69. 70. 71. 72. 73. 74. 75. 76. 77. 78. 79. 80. 81. 82. 83. 84. 85. 86. 87. 88. 89. 90. 91. 92. 93. 94. 95. 96. 97. 98. 99. 100. 101. 102. 103. 104. 105. 106. 107. 108. 109. 110. 111. 112. 113. 114. 115. 116. 117. 118. 119. 120. 121. 122. 123. 124. 125. 126. 127. 128. 129. 130. 131. 132. 133. 134. 135. 136. 137. 138. 139. 140. 141. 142. 143. 144. 145. 146. 147. 148. 149. 150. 151. 152. 153. 154. 155. 156. 157. 158. 159. 160. 161. 162. 163. 164. 165. 166. 167. 168. 169. 170. 171. 172. 173. 174. 175. 176. 177. 178. 179. 180. 181. 182. 183. 184. 185. 186. 187. 188. 189. 190. 191. 192. 193. 194. 195. 196. 197. 198. 199. 200. 201. 202. 203. 204. 205. 206. 207. 208. 209. 210. 211. 212. 213. 214. 215. 216. 217. 218. 219. 220. 221. 222. 223. 224. 225. 226. 227. 228. 229. 230. 231. 232. 233. 234. 235. 236. 237. 238. 239. 240. 241. 242. 243. 244. 245. 246. 247. 248. 249. 250. 251. 252. 253. 254. 255. 256. 257. 258. 259. 260. 261. 262. 263. 264. 265. 266. 267. 268. 269. 270. 271. 272. 273. 274. 275. 276. 277. 278. 279. 280. 281. 282. 283. 284. 285. 286. 287. 288. 289. 290. 291. 292. 293. 294. 295. 296. 297. 298. 299. 300. 301. 302. 303. 304. 305. 306. 307. 308. 309. 310. 311. 312. 313. 314. 315. 316. 317. 318. 319. 320. 321. 322. 323. 324. 325. 326. 327. 328. 329. 330. 331. 332. 333. 334. 335. 336. 337. 338. 339. 340. 341. 342. 343. 344. 345. 346. 347. 348. 349. 350. 351. 352. 353. 354. 355. 356. 357. 358. 359. 360. 361. 362. 363. 364. 365. 366. 367. 368. 369. 370. 371. 372. 373. 374. 375. 376. 377. 378. 379. 380. 381. 382. 383. 384. 385. 386. 387. 388. 389. 390. 391. 392. 393. 394. 395. 396. 397. 398. 399. 400. 401. 402. 403. 404. 405. 406. 407. 408. 409. 410. 411. 412. 413. 414. 415. 416. 417. 418. 419. 420. 421. 422. 423. 424. 425. 426. 427. 428. 429. 430. 431. 432. 433. 434. 435. 436. 437. 438. 439. 440. 441. 442. 443. 444. 445. 446. 447. 448. 449. 450. 451. 452. 453. 454. 455. 456. 457. 458. 459. 460. 461. 462. 463. 464. 465. 466. 467. 468. 469. 470. 471. 472. 473. 474. 475. 476. 477. 478. 479. 480. 481. 482. 483. 484. 485. 486. 487. 488. 489. 490. 491. 492. 493. 494. 495. 496. 497. 498. 499. 500. 501. 502. 503. 504. 505. 506. 507. 508. 509. 510. 511. 512. 513. 514. 515. 516. 517. 518. 519. 520. 521. 522. 523. 524. 525. 526. 527. 528. 529. 530. 531. 532. 533. 534. 535. 536. 537. 538. 539. 540. 541. 542. 543. 544. 545. 546. 547. 548. 549. 550. 551. 552. 553. 554. 555. 556. 557. 558. 559. 560. 561. 562. 563. 564. 565. 566. 567. 568. 569. 570. 571. 572. 573. 574. 575. 576. 577. 578. 579. 580. 581. 582. 583. 584. 585. 586. 587. 588. 589. 590. 591. 592. 593. 594. 595. 596. 597. 598. 599. 600. 601. 602. 603. 604. 605. 606. 607. 608. 609. 610. 611. 612. 613. 614. 615. 616. 617. 618. 619. 620. 621. 622. 623. 624. 625. 626. 627. 628. 629. 630. 631. 632. 633. 634. 635. 636. 637. 638. 639. 640. 641. 642. 643. 644. 645. 646. 647. 648. 649. 650. 651. 652. 653. 654. 655. 656. 657. 658. 659. 660. 661. 662. 663. 664. 665. 666. 667. 668. 669. 670. 671. 672. 673. 674. 675. 676. 677. 678. 679. 680. 681. 682. 683. 684. 685. 686. 687. 688. 689. 690. 691. 692. 693. 694. 695. 696. 697. 698. 699. 700. 701. 702. 703. 704. 705. 706. 707. 708. 709. 710. 711. 712. 713. 714. 715. 716. 717. 718. 719. 720. 721. 722. 723. 724. 725. 726. 727. 728. 729. 730. 731. 732. 733. 734. 735. 736. 737. 738. 739. 740. 741. 742. 743. 744. 745. 746. 747. 748. 749. 750. 751. 752. 753. 754. 755. 756. 757. 758. 759. 760. 761. 762. 763. 764. 765. 766. 767. 768. 769. 770. 771. 772. 773. 774. 775. 776. 777. 778. 779. 780. 781. 782. 783. 784. 785. 786. 787. 788. 789. 790. 791. 792. 793. 794. 795. 796. 797. 798. 799. 800. 801. 802. 803. 804. 805. 806. 807. 808. 809. 810. 811. 812. 813. 814. 815. 816. 817. 818. 819. 820. 821. 822. 823. 824. 825. 826. 827. 828. 829. 830. 831. 832. 833. 834. 835. 836. 837. 838. 839. 840. 84

1991

*1. 23 44 22 11 0000

[illegible]

$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x) \delta(x-a) dx = f(a)$

1. *Phragmites australis* (Cav.) Trin. ex Steud.

1. *Journal of the American Medical Association*, 1990; 263: 1025-1028.

$$(10) \quad \sum_{p=1}^q (2p-1) = q^2 \Rightarrow \sum_{p=1}^{q+1} (2p-1) = (q+1)^2 \quad [(9); *(4)]$$

$$(11) \quad \forall_n \left[\sum_{p=1}^n (2p-1) = n^2 \Rightarrow \sum_{p=1}^{n+1} (2p-1) = (n+1)^2 \right] \quad [(4) - (10)]$$

Part (iii):

$$(12) \quad \forall_n \sum_{p=1}^n (2p-1) = n^2 \quad [(3), (11), \text{PMI}]$$

*

[The fact that part (ii) of this proof is one step shorter than part (ii) of the proof given on pages 8-10 and 8-11 is wholly due to the fact that the instance:

$$(q+1)^2 = q^2 + [2(q+1) - 1] \quad [\text{cf. step (9) on page 8-11}]$$

of the theorem:

$$\forall_n (n+1)^2 = n^2 + [2(n+1) - 1] \quad [\text{cf. step (8) on page 8-11}]$$

is established directly in steps (7) - (9) of the proof above and that steps (6) - (9) of this proof obviate the explicit use of the substitution rule for equations which results in step (10) on page 8-11. "Doing the algebra" in the proof will result in column proofs of varying lengths. But, as the discussion of Theorem 130 brings out, replacing this by "theorem, instance, and substitution-step" results in proofs for which parts (ii) invariably consist of 9 steps.]

*

Paragraph proof for Part A:

(i) By the recursive definition,

$$\sum_{p=1}^1 (2p-1) = 2 \cdot 1 - 1 = 2 - 1 = 1 = 1^2.$$

(ii) Suppose that [for some integer q] $\sum_{p=1}^q (2p-1) = q^2$. Since, by the recursive definition,

1. The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \frac{1}{2} \left(1 + \sqrt{1 - 4x} \right) \quad (1)$$

for $x \in [0, 1]$. It is shown that the function $f(x)$ is continuous and strictly increasing on the interval $[0, 1]$.

2. In the second part of the paper, the properties of the function $f(x)$ are studied for $x \in [0, 1]$.

3. The third part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \frac{1}{2} \left(1 + \sqrt{1 - 4x} \right) \quad (2)$$

for $x \in [0, 1]$. It is shown that the function $f(x)$ is continuous and strictly increasing on the interval $[0, 1]$.

4. In the fourth part of the paper, the properties of the function $f(x)$ are studied for $x \in [0, 1]$.

5. The fifth part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \frac{1}{2} \left(1 + \sqrt{1 - 4x} \right) \quad (3)$$

for $x \in [0, 1]$. It is shown that the function $f(x)$ is continuous and strictly increasing on the interval $[0, 1]$.

6. In the sixth part of the paper, the properties of the function $f(x)$ are studied for $x \in [0, 1]$.

7. The seventh part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \frac{1}{2} \left(1 + \sqrt{1 - 4x} \right) \quad (4)$$

for $x \in [0, 1]$. It is shown that the function $f(x)$ is continuous and strictly increasing on the interval $[0, 1]$.

8. In the eighth part of the paper, the properties of the function $f(x)$ are studied for $x \in [0, 1]$.

9. The ninth part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \frac{1}{2} \left(1 + \sqrt{1 - 4x} \right) \quad (5)$$

for $x \in [0, 1]$. It is shown that the function $f(x)$ is continuous and strictly increasing on the interval $[0, 1]$.

10. In the tenth part of the paper, the properties of the function $f(x)$ are studied for $x \in [0, 1]$.

11. The eleventh part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \frac{1}{2} \left(1 + \sqrt{1 - 4x} \right) \quad (6)$$

$$\sum_{p=1}^{q+1} (2p-1) = \sum_{p=1}^q (2p-1) + [2(q+1)-1],$$

it follows that

$$\begin{aligned} \sum_{p=1}^{q+1} (2p-1) &= q^2 + [2(q+1)-1] \\ &= q^2 + 2q + 2 - 1 = (q+1)^2. \end{aligned}$$

Hence,

$$\sum_{p=1}^q (2p-1) = q^2 \Rightarrow \sum_{p=1}^{q+1} (2p-1) = (q+1)^2.$$

Consequently,

$$\forall_n \left[\sum_{p=1}^n (2p-1) = n^2 \Rightarrow \sum_{p=1}^{n+1} (2p-1) = (n+1)^2 \right].$$

(iii) So, in view of (i) and (ii), it follows, by induction, that

$$\forall_n \sum_{p=1}^n (2p-1) = n^2.$$

*

Answers for Part B.

1. Column proof:

Part (i):

$$(1) \quad \sum_{p=1}^1 (5+4p) = 5 + 4 \cdot 1 \quad [\text{recursive definition}]$$

$$(2) \quad 1(2 \cdot 1 + 7) = 5 + 4 \cdot 1 \quad [\text{theorem}]$$

$$(3) \quad \sum_{p=1}^1 (5+4p) = 1(2 \cdot 1 + 7) \quad [(1), (2)]$$

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Part (ii):

$$(4) \quad \sum_{p=1}^q (5+4p) = q(2q+7) \quad [\text{inductive hypothesis}]^*$$

$$(5) \quad \forall_n \sum_{p=1}^{n+1} (5+4p) = \sum_{p=1}^n (5+4p) + [5+4(n+1)] \quad [\text{recursive definition}]$$

$$(6) \quad \sum_{p=1}^{q+1} (5+4p) = \sum_{p=1}^q (5+4p) + [5+4(q+1)] \quad [(5)]$$

$$(7) \quad = q(2q+7) + [5+4(q+1)] \quad [(4), (6)]$$

$$(8) \quad = 2q^2 + 7q + 5 + 4q + 4 \quad \left. \begin{array}{l} (9) \quad = 2q^2 + 11q + 9 \\ (10) \quad = (q+1)(2q+9) \\ (11) \quad = (q+1)[2(q+1)+7] \end{array} \right\} \quad [\text{algebra}]$$

$$(12) \quad \sum_{p=1}^q (5+4p) = q(2q+7) \Rightarrow \sum_{p=1}^{q+1} (5+4p) = (q+1)[2(q+1)+7] \quad [(11); *(4)]$$

$$(13) \quad \forall_n \sum_{p=1}^n (5+4p) = n(2n+7) \Rightarrow \sum_{p=1}^{n+1} (5+4p) = (n+1)[2(n+1)+7] \quad [(4) - (12)]$$

Part (iii):

$$(14) \quad \forall_n \sum_{p=1}^n (5+4p) = n(2n+7) \quad [(3), (13), \text{PMI}]$$

Paragraph proof [for Exercise 1 of Part B]:

(i) By the recursive definition,

$$\sum_{p=1}^1 (5 + 4p) = 5 + 4 \cdot 1 = 9 = 1 \cdot (2 \cdot 1 + 7).$$

(ii) Suppose that $\sum_{p=1}^q (5 + 4p) = q(2q + 7)$. Since, by the recursive definition,

$$\sum_{p=1}^{q+1} (5 + 4p) = \sum_{p=1}^q (5 + 4p) + [5 + 4(q + 1)],$$

it follows that

$$\begin{aligned} \sum_{p=1}^{q+1} (5 + 4p) &= q(2q + 7) + [5 + 4(q + 1)] \\ &= 2q^2 + 7q + 5 + 4q + 4 \\ &= 2q^2 + 11q + 9 \\ &= (q + 1)(2q + 9) \\ &= (q + 1)[2(q + 1) + 7]. \end{aligned}$$

Hence,

$$\sum_{p=1}^q (5 + 4p) = q(2q + 7) \Rightarrow \sum_{p=1}^{q+1} (5 + 4p) = (q + 1)[2(q + 1) + 7].$$

Consequently,

$$\forall_n \left[\sum_{p=1}^n (5 + 4p) = n(2n + 7) \Rightarrow \sum_{p=1}^{n+1} (5 + 4p) = (n + 1)[2(n + 1) + 7] \right].$$

(iii) So, in view of (i) and (ii), it follows from the PMI that

$$\forall_n \sum_{p=1}^n (5 + 4p) = n(2n + 7).$$

*

Since the styles of the proofs for Exercises 2, 3, and 4 are precisely the same as the styles for the proofs for Exercise 1, we give below just the "algebra" parts for these exercises.

*

$$\begin{aligned} 2. \quad \sum_{p=1}^{q+1} (3p - 2) &= \sum_{p=1}^q (3p - 2) + [3(q + 1) - 2] \\ &= \frac{q(3q - 1)}{2} + [3(q + 1) - 2] \\ &= \frac{q(3q - 1) + 6(q + 1) - 4}{2} \\ &= \frac{3q^2 - q - 4 + 6(q + 1)}{2} \\ &= \frac{(3q - 4)(q + 1) + 6(q + 1)}{2} \\ &= \frac{(q + 1)(3q - 4 + 6)}{2} \\ &= \frac{(q + 1)[3(q + 1) - 1]}{2} \end{aligned}$$

$$\begin{aligned}
3. \quad \sum_{p=1}^{q+1} (2p - 1)^2 &= \sum_{p=1}^q (2p - 1)^2 + [2(q + 1) - 1]^2 \\
&= \frac{q(2q - 1)(2q + 1)}{3} + [2(q + 1) - 1]^2 \\
&= \frac{q(2q - 1)(2q + 1) + 3(2q + 1)^2}{3} \\
&= \frac{(2q + 1)[q(2q - 1) + 3(2q + 1)]}{3} \\
&= \frac{(2q + 1)(2q^2 - q + 6q + 3)}{3} \\
&= \frac{(2q + 1)(2q^2 + 5q + 3)}{3} \\
&= \frac{(2q + 1)(q + 1)(2q + 3)}{3} \\
&= \frac{(q + 1)[2(q + 1) - 1][2(q + 1) + 1]}{3}
\end{aligned}$$

$$\star 4. \quad \sum_{p=1}^{q+1} \frac{1}{p^2} = \sum_{p=1}^q \frac{1}{p^2} + \frac{1}{(q+1)^2}$$

$$\leq 2 - \frac{1}{q} + \frac{1}{(q+1)^2}$$

$$= 2 - \left(\frac{1}{q} - \frac{1}{(q+1)^2} \right)$$

$$= 2 - \frac{q^2 + q + 1}{q(q+1)^2}$$

$$= 2 - \frac{q(q+1) + 1}{q(q+1)^2} \left. \vphantom{\frac{q(q+1) + 1}{q(q+1)^2}} \right\}$$

$$< 2 - \frac{q(q+1)}{q(q+1)^2}$$

$$= 2 - \frac{1}{q+1}$$

$$\begin{array}{ccc} \boxed{\frac{q(q+1)+1}{q(q+1)^2}} & & \boxed{\frac{q(q+1)}{q(q+1)^2}} \\ \downarrow & & \downarrow \\ \forall_x \forall_y \forall_z [z > y \Rightarrow x - z < x - y] \\ \uparrow & & \uparrow \\ 2 & & 2 \end{array}$$

[The theorem used in getting from the fifth line to the sixth line is both obvious and frequently useful. Its proof is a good review exercise for students:

Suppose that $c > b$. Then, by Theorem 94, $-c < -b$ and, by the atpi and the cpa, $a + -c < a + -b$. So, by the ps, $a - c < a - b$. Hence, if $c > b$ then $a - c < a - b$. Consequently, $\forall_x \dots \dots]$

1. The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \sum_{n=0}^{\infty} \frac{f_n(x)}{n!}$, where $f_n(x)$ are the solutions of the system of equations $f_n'(x) = -f_n(x) + f_{n+1}(x)$, $f_0(x) = 1$. It is shown that $f(x)$ is a solution of the differential equation $f'(x) = -f(x) + f(x+1)$.

$$\begin{aligned}
 f(x) &= \sum_{n=0}^{\infty} \frac{f_n(x)}{n!} \\
 f_n(x) &= \sum_{k=0}^n \frac{f_k(x)}{k!} \frac{f_{n-k}(x)}{(n-k)!}
 \end{aligned}$$

2. The second part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \sum_{n=0}^{\infty} \frac{f_n(x)}{n!}$, where $f_n(x)$ are the solutions of the system of equations $f_n'(x) = -f_n(x) + f_{n+1}(x)$, $f_0(x) = 1$. It is shown that $f(x)$ is a solution of the differential equation $f'(x) = -f(x) + f(x+1)$.

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$$\frac{f(x)}{f(x+1)} = \sum_{n=0}^{\infty} \frac{f_n(x)}{n!} \frac{f_{n+1}(x)}{(n+1)!}$$

$$\frac{f(x)}{f(x+1)} = \sum_{n=0}^{\infty} \frac{f_n(x)}{n!} \frac{f_{n+1}(x)}{(n+1)!}$$

$$\frac{f(x)}{f(x+1)} = \sum_{n=0}^{\infty} \frac{f_n(x)}{n!} \frac{f_{n+1}(x)}{(n+1)!}$$

$$\frac{f(x)}{f(x+1)} = \sum_{n=0}^{\infty} \frac{f_n(x)}{n!} \frac{f_{n+1}(x)}{(n+1)!}$$

[There are further generalizations which students will discover if they solve the exercises of Part ☆D on pages 8-56 and 8-57.]

The algebra steps in the proof of the generalization are:

$$\begin{aligned}
 \sum_{p=1}^{q+1} \frac{1}{[mp - (m-1)](mp+1)} &= \sum_{p=1}^q \frac{1}{[mp - (m-1)](mp+1)} \\
 &\quad + \frac{1}{[m(q+1) - (m-1)][m(q+1)+1]} \\
 &= \frac{q}{mq+1} + \frac{1}{[m(q+1) - (m-1)][m(q+1)+1]} \\
 &= \frac{q}{mq+1} + \frac{1}{(mq+1)[m(q+1)+1]} \\
 &= \frac{mq^2 + (m+1)q + 1}{(mq+1)[m(q+1)+1]} \\
 &= \frac{(mq+1)(q+1)}{(mq+1)[m(q+1)+1]} \\
 &= \frac{q+1}{m(q+1)+1}
 \end{aligned}$$

PHYSICS 311

LECTURE 10

LECTURE 10: THE HARMONIC OSCILLATOR

1. INTRODUCTION

The harmonic oscillator is a fundamental model in physics. It describes the motion of a mass attached to a spring. The potential energy is given by $V(x) = \frac{1}{2}kx^2$, where k is the spring constant and x is the displacement from equilibrium. The total energy is conserved, and the motion is periodic.

2. CLASSICAL MECHANICS

3. QUANTUM MECHANICS

4. THE HARMONIC OSCILLATOR

5. THE HARMONIC OSCILLATOR

6. THE HARMONIC OSCILLATOR

7. THE HARMONIC OSCILLATOR

*

Answer for Part ★E.

The two theorems:

$$\forall_n \sum_{p=1}^n \frac{1}{p(p+1)} = \frac{n}{n+1}$$

$$\forall_n \sum_{p=1}^n \frac{1}{(2p-1)(2p+1)} = \frac{n}{2n+1}$$

bear a family likeness. One feels that there must be a summation theorem of the form:

$$\forall_n \sum_{p=1}^n \frac{1}{(3p-1)(3p+1)} = \frac{n}{3n+1}$$

and, in fact, that there must be a generalization of which these theorems are the first three instances. To discover this generalization, we begin by trying to make the first two theorems look as alike as possible:

$$\forall_n \sum_{p=1}^n \frac{1}{(1p-0)(1p+1)} = \frac{n}{1n+1}$$

$$\forall_n \sum_{p=1}^n \frac{1}{(2p-1)(2p+1)} = \frac{n}{2n+1}$$

Surely, the third theorem in the family must be:

$$\forall_n \sum_{p=1}^n \frac{1}{(3p-2)(3p+1)} = \frac{n}{3n+1}$$

and the generalization we seek must be:

$$\forall_m \forall_n \sum_{p=1}^n \frac{1}{[mp - (m-1)](mp+1)} = \frac{n}{mn+1}$$

Figure 1. The effect of the concentration of the initiator on the polymerization of α -methylstyrene in the presence of $\text{Cu}(\text{NO}_3)_2 \cdot 3\text{H}_2\text{O}$ at 50°C .

$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$

[illegible]

1. *Journal of the American Medical Association*, 1990; 263: 1025-1028.

2. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$$\sum_{p=1}^3 a_p = \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} = \frac{2}{5} + \frac{1}{35} = \frac{3}{7}$$

$$\sum_{p=1}^4 a_p = \frac{4}{9} \quad \sum_{p=1}^5 a_p = \frac{5}{11} \quad \sum_{p=1}^6 a_p = \frac{6}{13}$$

It appears that $\forall_n \sum_{p=1}^n a_p = \frac{n}{2n+1}$.

The next step is to give an inductive proof. To save space here in the COMMENTARY we give just the algebra steps:

$$\begin{aligned} \sum_{p=1}^{q+1} \frac{1}{(2p-1)(2p+1)} &= \sum_{p=1}^q \frac{1}{(2p-1)(2p+1)} + \frac{1}{[2(q+1)-1][2(q+1)+1]} \\ &= \frac{q}{2q+1} + \frac{1}{(2q+1)[2(q+1)+1]} \\ &= \frac{2q^2 + 3q + 1}{(2q+1)[2(q+1)+1]} \\ &= \frac{q+1}{2(q+1)+1} \end{aligned}$$

Finally, we prove the generalization in question--that is [by substitution], we prove:

$$\begin{aligned} \forall_n \frac{n}{2n+1} &< \frac{1}{2} \\ \frac{q}{2q+1} &< \frac{1}{2} \iff 2q < 2q+1 \\ &\iff 0 < 1 \end{aligned}$$

So, since $0 < 1$, it follows that $\frac{q}{2q+1} < \frac{1}{2}$. Consequently, $\forall_n \dots$

3. One such general theorem is:

$$\forall_x \forall_n \sum_{p=1}^n [xp - (x-1)] = \frac{n[xn - (x-2)]}{2}$$

Another is:

$$\forall_y \forall_x \forall_n \sum_{p=1}^n (xp + y) = \frac{n[xn + y + (x+y)]}{2}$$

The proofs of these theorems are entirely analogous to those for the theorems in Exercise 2 of Part B and in Exercise 2 of Part C. However, each involves an additional generalizing step after part (iii).

*

Answer for Part D.

The second problem on page 8-1 may have prepared students for this exercise. However, it is probably still necessary to do a bit more preliminary work to get students started. We have here a sequence which starts

$$\frac{1}{1 \cdot 3}, \frac{1}{3 \cdot 5}, \frac{1}{5 \cdot 7}, \frac{1}{7 \cdot 9}, \dots$$

and our problem is to show that, no matter how many terms we take, starting with the first, we can never get a continued sum which is as large as $\frac{1}{2}$. As indicated in the hint, we compute some of the terms in the continued sum sequence.

$$\sum_{p=1}^1 a_p = \frac{1}{1 \cdot 3} = \frac{1}{3}$$

$$\sum_{p=1}^2 a_p = \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} = \frac{1}{3} + \frac{1}{15} = \frac{2}{5}$$

Answers for Part C.

$$1. \sum_{p=1}^{100} (3p - 2) = \frac{100(3 \cdot 100 - 1)}{2} = 14950$$

2. A procedure to use in discovering the theorems in Exercise 1 and in the present exercise is the procedure illustrated in the following diagrams:

$$\begin{array}{c}
 1 + 4 + 7 + 10 + 13 + 16 = 6 \times \frac{17}{2} = 6 \times \frac{16+1}{2} = \frac{6[(3 \cdot 6 - 2) + 1]}{2} \\
 \swarrow \quad \nwarrow \quad \nearrow \quad \nwarrow \quad \nearrow \quad \nearrow \\
 17 \quad 17 \quad 17 \\
 \searrow \quad \swarrow \quad \swarrow \quad \searrow \quad \swarrow \quad \searrow \\
 17 \quad 17 \quad 17
 \end{array}
 = \frac{6(3 \cdot 6 - 1)}{2}$$

$$\begin{array}{c}
 1 + 5 + 9 + 13 + 17 + 21 = 6 \times \frac{22}{2} = 6 \times \frac{21+1}{2} = \frac{6[(4 \cdot 6 - 3) + 1]}{2} \\
 \swarrow \quad \nwarrow \quad \nearrow \quad \nwarrow \quad \nearrow \quad \nearrow \\
 22 \quad 22 \quad 22 \\
 \searrow \quad \swarrow \quad \swarrow \quad \searrow \quad \swarrow \quad \searrow \\
 22 \quad 22 \quad 22
 \end{array}
 = \frac{6(4 \cdot 6 - 2)}{2}$$

This suggests for Exercise 2 the theorem:

$$\forall_n \sum_{p=1}^n (4p - 3) = \frac{n(4n - 2)}{2} = n(2n - 1)$$

The algebra steps in the inductive proof of this theorem are:

$$\begin{aligned}
 \sum_{p=1}^{q+1} (4p - 3) &= \sum_{p=1}^q (4p - 3) + [4(q+1) - 3] \\
 &= q(2q - 1) + [4(q+1) - 3] \\
 &= 2q^2 - q - 3 + 4(q+1) \\
 &= (2q - 3)(q+1) + 4(q+1) \\
 &= (q+1)(2q+1) \\
 &= (q+1)[2(q+1) - 1]
 \end{aligned}$$

Answers for Part F.

1. The sum is $1^2 + 3^2 + 5^2 + \dots + 29^2$, that is, $\sum_{p=1}^{15} (2p - 1)^2$.

The theorem in Exercise 3 of Part B on page 8-12 is relevant here.

$$\sum_{p=1}^{15} (2p - 1)^2 = \frac{15(2 \cdot 15 - 1)(2 \cdot 15 + 1)}{3} = \frac{15 \cdot 29 \cdot 31}{3} = 4495$$

2. In this exercise we are looking for $4 \cdot 1 + 4 \cdot 3 + 4 \cdot 5 + \dots + 4 \cdot 29$,

that is, $4(1 + 3 + 5 + \dots + 29)$, or $4 \sum_{p=1}^{15} (2p - 1)$.

The theorem in Exercise A on page 8-12 applies here.

$$4 \sum_{p=1}^{15} (2p - 1) = 4 \cdot (15)^2 = 900$$

*

Answer for Part G.

The purpose of this exercise is to test the student's understanding of what a continued sum sequence is. Given a sequence a , it is possible to compute the terms of its continued sum sequence. If b is the continued sum sequence, then

$$b_1 = a_1, \quad b_2 = a_1 + a_2, \quad b_3 = a_1 + a_2 + a_3, \quad \text{etc.}$$

$$\text{Now, } a_1 = \frac{1}{1 \cdot 2 \cdot 3} = \frac{1}{6}, \quad a_2 = \frac{1}{24}, \quad a_3 = \frac{1}{60}, \quad a_4 = \frac{1}{120}, \quad \dots$$

$$\text{So, } b_1 = \frac{1}{6}, \quad b_2 = \frac{5}{24}, \quad b_3 = \frac{9}{40}, \quad b_4 = \frac{7}{30}, \quad \dots$$

We now examine each of the six given possibilities to see which can be eliminated.

In the case of (a), we find, by computation, that $b_1 = \frac{1}{6}$, $b_2 = \frac{5}{24}$, and $b_3 = \frac{7}{30}$. Since $\frac{7}{30} \neq \frac{9}{40}$, the sequence in (a) cannot be the continued sum sequence.

In the case of (b), $b_1 = \frac{1}{6}$, $b_2 = \frac{5}{24}$, $b_3 = \frac{9}{40}$, and $b_4 = \frac{11}{48}$. Since $\frac{11}{48} \neq \frac{7}{30}$, (b) is out.

Students will find that the sequence of part (c) gives the correct values for the continued sum sequence for sequence a as far as they have patience to check. They can prove that it is the continued sum sequence by showing that, for each p , $b_p - b_{p-1} = a_p$:

$$\begin{aligned} \frac{p(p+3)}{4(p+1)(p+2)} - \frac{(p-1)(p+2)}{4p(p+1)} &= \frac{p^2(p+3) - (p-1)(p+2)^2}{4p(p+1)(p+2)} \\ &= \frac{4}{4p(p+1)(p+2)} \\ &= \frac{1}{p(p+1)(p+2)} \end{aligned}$$

So, the sequence in (c) is the continued sum sequence.

In the case of (d), $b_1 = \frac{3}{32}$. Since $\frac{3}{32} \neq \frac{1}{6}$, (d) is out.

In the case of (e), $b_1 = \frac{1}{6}$ and $b_2 = \frac{1}{3}$. Since $\frac{1}{3} \neq \frac{5}{24}$, (e) is out.

The sequence in (f) is the continued sum sequence. It is defined recursively in precisely the way in which the continued sum sequence for a is defined recursively. This is an introduction to the discussion which follows and which leads up to Theorem 130. In this connection you may wish to re-read the COMMENTARY for pages 7-63 through 7-65 of Unit 7. [In that COMMENTARY it is shown that at most one sequence satisfies a given recursive definition.]

The applicability of Theorem 130 [mentioned in lines 8 and 9] can be made more apparent by doing the following with the recursive definition of T:

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} \nearrow \\ b_1 \end{array} & & \begin{array}{c} \nwarrow \\ a_1 \end{array} \\
 & T_1 = 1 & \\
 \end{array} \\
 \\
 \forall_n \underbrace{T_{n+1}}_{\begin{array}{c} \uparrow \\ b_{n+1} \end{array}} = \underbrace{T_n}_{\begin{array}{c} \uparrow \\ b_n \end{array}} + \underbrace{(n+1)}_{\begin{array}{c} \uparrow \\ a_{n+1} \end{array}}
 \end{array}$$

*

Here is the algebra needed in proving the theorem (ii):

$$\begin{aligned}
 & \frac{n(n+3)}{4(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} \\
 &= \frac{n(n+3)(n+3) + 4}{4(n+1)(n+2)(n+3)} \\
 &= \frac{n^3 + 6n^2 + 9n + 4}{4(n+1)(n+2)(n+3)} \\
 &= \frac{(n+1)(n^2 + 5n + 4)}{4(n+1)(n+2)(n+3)} \quad \left. \begin{array}{l} \nearrow \\ \nwarrow \end{array} \right\} \begin{array}{l} \text{via the division-with-} \\ \text{remainder algorithm} \end{array} \\
 &= \frac{(n+1)(n+4)}{4(n+2)(n+3)}
 \end{aligned}$$

1. The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function.

$$f(x) = \int_0^x f(t) dt$$

2. The second part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function.

$$f(x) = \int_0^x f(t) dt$$

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3. The third part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function.

$$f(x) = \int_0^x f(t) dt$$

$$f(x) = \int_0^x f(t) dt$$

$$3. \frac{1^2}{(2 \cdot 1 - 1)(2 \cdot 1 + 1)} = \frac{1}{1 \cdot 3} = \frac{1}{3}; \quad \frac{1(1+1)}{2(2 \cdot 1 + 1)} = \frac{2}{6} = \frac{1}{3}$$

$$b_{q+1} = \frac{(q+1)(q+2)}{2(2q+3)};$$

$$\begin{aligned} b_q + a_{q+1} &= \frac{q(q+1)}{2(2q+1)} + \frac{(q+1)^2}{(2q+1)(2q+3)} = \frac{q(q+1)(2q+3) + 2(q+1)^2}{2(2q+1)(2q+3)} \\ &= \frac{(q+1)(2q^2 + 3q + 2q + 2)}{2(2q+1)(2q+3)} = \frac{(q+1)(2q+1)(q+2)}{2(2q+1)(2q+3)} \\ &= \frac{(q+1)(q+2)}{2(2q+3)} \end{aligned}$$

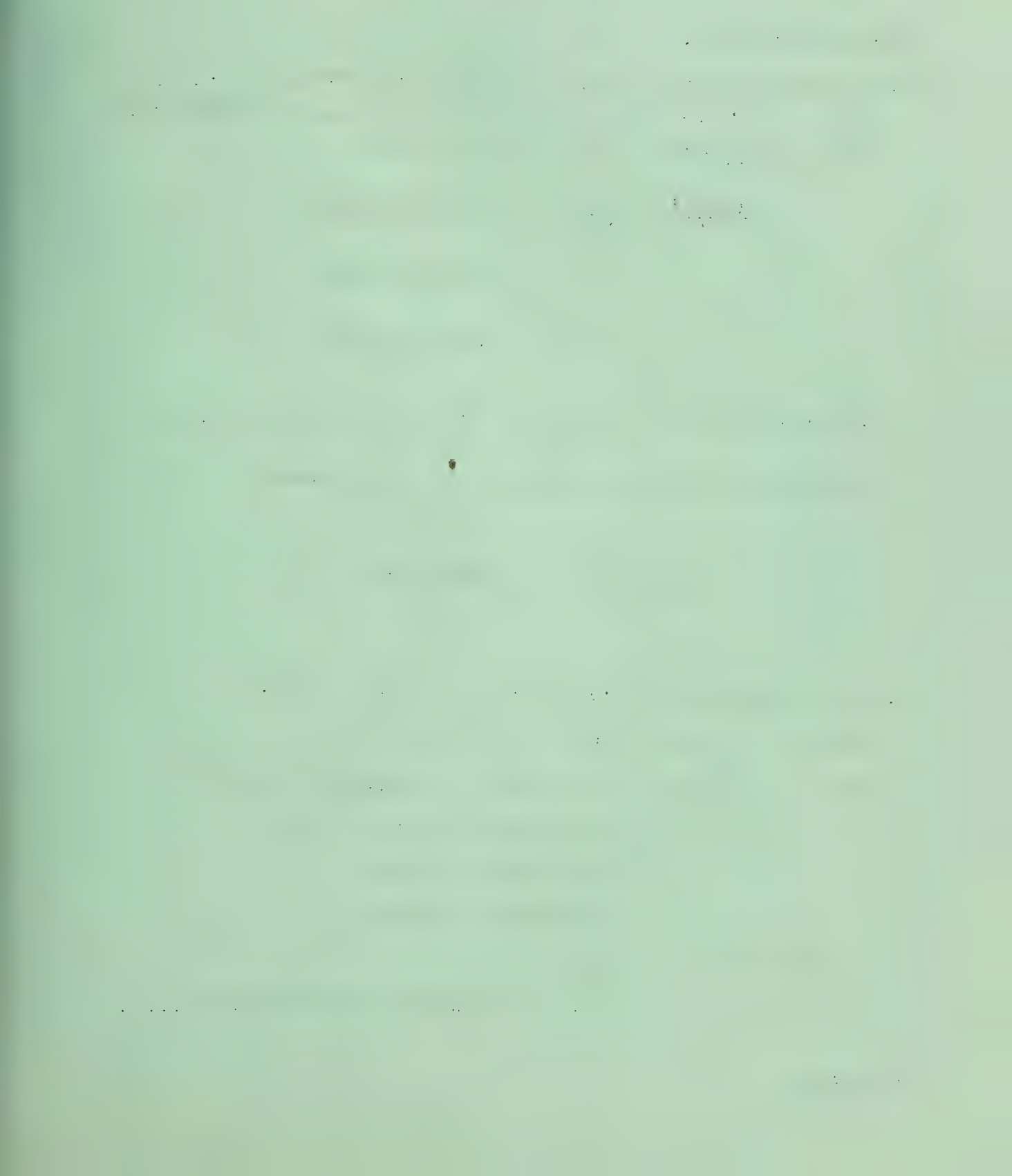
*

Answers for Part B.

1.

| n | 1 | 2 | 3 | 4 | 5 |
|--------------------|---|---|----|-----|-----|
| $\sum_{p=1}^n p^2$ | 1 | 5 | 14 | 30 | 55 |
| $\sum_{p=1}^n p$ | 1 | 3 | 6 | 10 | 15 |
| $\sum_{p=1}^n p^3$ | 1 | 9 | 36 | 100 | 225 |

Exercise 2 of Part B is discussed in the COMMENTARY for page 8-20.



Answers for Part A.

1. In this case $a_p = 3p + 5$ and $b_p = p(3p + 13)/2$. Now, $a_1 = 3 \cdot 1 + 5 = 8$, and $b_1 = 1(3 \cdot 1 + 13)/2 = 8$. So, $a_1 = b_1$. Also, $b_{q+1} = (q+1)(3q+16)/2$, and $b_q + a_{q+1} = q(3q+13)/2 + (3q+8)$. Since

$$\begin{aligned} \frac{q(3q+13)}{2} + (3q+8) &= \frac{3q^2 + 13q + 6q + 16}{2} \\ &= \frac{3q^2 + 19q + 16}{2} \\ &= \frac{(q+1)(3q+16)}{2} \end{aligned}$$

it follows that $b_{q+1} = b_q + a_{q+1}$. Hence, $\forall_n b_{n+1} = b_n + a_{n+1}$.

Consequently, by Theorem 130, $\forall_n \sum_{p=1}^n a_p = b_n$ -- that is,

$$\forall_n \sum_{p=1}^n (3p+5) = \frac{n(3n+13)}{2}.$$

2. $a_1 = b_1$ because $a_1 = 6 \cdot 1^2 + 12 \cdot 1 = 18$, and $b_1 = 1 \cdot 2 \cdot 9 = 18$.

Since $b_{q+1} = (q+1)(q+2)(2q+9)$

and $b_q + a_{q+1} = q(q+1)(2q+7) + [6(q+1)^2 + 12(q+1)]$

$$\begin{aligned} &= (q+1)(2q^2 + 7q + 6q + 6 + 12) \\ &= (q+1)(2q^2 + 13q + 18) \\ &= (q+1)(q+2)(2q+9), \end{aligned}$$

it follows that $b_{q+1} = b_q + a_{q+1}$.

So, $\forall_n b_{n+1} = b_n + a_{n+1}$. Consequently, by Theorem 130,

2. (a) Notice:

$$\frac{\sum_{p=1}^1 p^2}{\sum_{p=1}^1 p} = \frac{1}{1} = \frac{3}{3}; \quad \frac{\sum_{p=1}^2 p^2}{\sum_{p=1}^2 p} = \frac{5}{3}; \quad \frac{\sum_{p=1}^3 p^2}{\sum_{p=1}^3 p} = \frac{14}{6} = \frac{7}{3}; \quad \frac{9}{3}; \quad \frac{11}{3}$$

It appears that

$$\forall_n \frac{\sum_{p=1}^n p^2}{\sum_{p=1}^n p} = \frac{2n+1}{3}.$$

So, we guess that the summation theorem for the sequence of squares is:

$$\forall_n \sum_{p=1}^n p^2 = \frac{n(n+1)(2n+1)}{6}$$

Next, notice that:

$$\sum_{p=1}^1 p^3 = \left(\sum_{p=1}^1 p \right)^2; \quad \sum_{p=1}^2 p^3 = \left(\sum_{p=1}^2 p \right)^2; \quad \sum_{p=1}^3 p^3 = \left(\sum_{p=1}^3 p \right)^2; \quad \dots$$

So, we guess that the summation theorem for the sequence of cubes is:

$$\forall_n \sum_{p=1}^n p^3 = \left[\frac{n(n+1)}{2} \right]^2$$

2. (b) Use mathematical induction directly, or use Theorem 130. In either case, the algebraic manipulation is the same. For the sequence of squares:

$$1^2 = 1 = \frac{1 \cdot 2 \cdot 3}{6};$$

$$\begin{aligned} \frac{q(q+1)(2q+1)}{6} + (q+1)^2 &= \frac{(q+1)(2q^2 + q + 6q + 6)}{6} \\ &= \frac{(q+1)(q+2)(2q+3)}{6} \end{aligned}$$

For the sequence of cubes:

$$1^3 = 1 = \left[\frac{1 \cdot 2}{2} \right]^2;$$

$$\begin{aligned} \left[\frac{q(q+1)}{2} \right]^2 + (q+1)^3 &= \frac{(q+1)^2(q^2 + 4q + 4)}{4} \\ &= \frac{(q+1)^2(q+2)^2}{4} = \left[\frac{(q+1)(q+2)}{2} \right]^2 \end{aligned}$$

*

Sums of powers of positive integers are discussed in two recent articles in The Mathematics Teacher:

"Sums of Powers of Integers" by Robert C. Yates, April 1959

"Some Challenging Series ..." by Robert M. Rippey, January 1961

See also:

"On the Sum of Powers of Natural Numbers" by J. G. Christiano, in the February 1961 issue of The American Mathematical Monthly.

The recursive procedure for finding summation theorems for sums of powers which is given in the first paper is explored in Part B on pages 8-54 and 8-55 and developed explicitly on pages 8-202 ff. The Fibonacci sequences discussed in the second paper are introduced in Part ☆E on

pages 8-24 and 8-25 and are studied further in Parts ☆D and ☆E on pages 8-46 and 8-47 and in Parts ☆G and ☆H on pages 8-126 through 8-128.

*

3. This exercise provides practice in applying the three theorems of Exercise 2.

$$(a) \sum_{p=1}^{10} p = \frac{10 \cdot 11}{2} = 55$$

$$(b) \sum_{p=1}^{1000} p^2 = \frac{1000 \cdot 1001 \cdot 2001}{6} = 333833500$$

$$(c) 2850$$

$$(d) 2850 - 1830 = 1020$$

$$(e) \frac{48 \cdot 49}{2} - \frac{39 \cdot 40}{2} = 4(6 \cdot 49 - 39 \cdot 5) = 396$$

$$(f) \frac{200 \cdot 201}{2} - \frac{100 \cdot 101}{2} = \frac{100}{2}(2 \cdot 201 - 101) = \frac{30100}{2} = 15050$$

$$(g) \sum_{p=1}^{1000} p^2 - \sum_{p=1}^{99} p^2 = \frac{1000 \cdot 1001 \cdot 2001}{6} - \frac{99 \cdot 100 \cdot 199}{6} \\ = 333833500 - 328350 = 333505150$$

$$(h) \sum_{p=1}^{1000} p^3 - \sum_{p=1}^{99} p^3 = \left(\frac{1000 \cdot 1001}{2} \right)^2 - \left(\frac{99 \cdot 100}{2} \right)^2 \\ = 250500250000 - 24502500 = 250475747500$$

(i) By the associative and commutative principles [or Theorem 5], the given expression is equivalent to:

$$\sum_{p=1}^{1000} p^2 + \sum_{p=1}^{1000} p^3$$

So [using computations in parts (g) and (h)], the result is $333833500 + 250500250000$, or 250834083500 .

(j) Since, for each n , $n^2(n + 1) = n^2 + n^3$, the answer to this exercise is that of part (i).

☆(k) By inspection, it is clear that the first factor of the p th term is p . This leaves one free to concentrate on the second factors of the first few terms--

2, 5, 10, 17, 26.

It appears from this that the second factor of the p th term is $p^2 + 1$. To check, and to find the number of terms involved, we look for positive integral solutions of ' $p(p^2 + 1) = 738$ '. Trying 6, 7, 8, and 9 shows that 738 is the 9th term of the sequence whose p th term is $p(p^2 + 1)$. [One can save half the labor by noticing that neither 7 nor 8 is a factor of 738.] So, the given expression is equivalent to:

$$\sum_{p=1}^9 p(p^2 + 1)$$

By the \mathcal{L} dpma and Theorem 5 [or Theorem 134], and the theorems of Exercise 2,

$$\begin{aligned} \sum_{p=1}^9 p(p^2 + 1) &= \sum_{p=1}^9 (p^3 + p) = \sum_{p=1}^9 p^3 + \sum_{p=1}^9 p \\ &= \left(\frac{9 \cdot 10}{2}\right)^2 + \frac{9 \cdot 10}{2} = 45 \cdot 46 = 2070 \end{aligned}$$

*

Exercise 4 provides preparation for Exercise 7 on page 8-22.

*

4. (a) 476776 [(976 · 977)/2 = 476776]

(b) 976 [Since the domain of ' n ' is I^+ , the equation

$$\frac{n(n + 1)}{2} = 476776 \text{ has only one root.}]$$

(c) 25 $\left[\frac{n(n+1)}{2} = 325 \right.$

$$\begin{aligned} &\iff n^2 + n - 650 = 0 \\ &\iff (n+26)(n-25) = 0 \\ &\iff n = 25 \end{aligned}$$

Note that the last ' \iff ' is justified by the fact that the domain of ' n ' is I^+ . Were the domain of ' n ' the set of all real numbers--or even the set of all integers--the final step would be ' $\iff (n = -26$ or $n = 25)$ '.]

(d) no solution [There is no positive integer n such that $n^2 + n - 1088 = 0$. In fact, using the quadratic formula, the discriminant of this equation is 4353. One can see that 4353 is not a square by noting that it is divisible by 3 but not by 3^2 .]

(e) 65 $\left[\frac{(n+1)(n+2)}{2} = 2211 \iff (n-65)(n+68) = 0 \right]$

(f) $\{n: n < 19\}$ $[n^2 + n - 380 < 0$

$$\begin{aligned} &\iff (n+20)(n-19) < 0 \\ &\iff n < 19 \end{aligned}$$

Note that the third line need not be ' $\iff -20 < n < 19$ ' because the domain of ' n ' is I^+ .]

Answers for the rest of Part B are in the COMMENTARY for page 8-21 and for page 8-22.

(c) With respect to the coordinate system described in (d), the slope of $\overrightarrow{A_1 A_2} = \frac{1 - 0.5}{2 - 0.5} = \frac{1}{3}$ and the slope of $\overrightarrow{A_2 A_3} = \frac{1.5 - 1}{4.5 - 2} = \frac{1}{5}$. So, since A_1 , A_2 , and A_3 are noncollinear, so are A_1 , A_2 , A_3 , ..., and A_{10} .

☆(d) We notice that the x-coordinate of P_n , for $n \leq 10$, is the sum of the first n positive integers. We see that the x-coordinate of A_n and of B_n is $\frac{1}{2}(P_n + P_{n-1})$. So [noting that the x-coordinate of P_0 is 0], for $n \leq 10$, the x-coordinate of A_n and of B_n is $\frac{1}{2}[n(n+1)/2 + (n-1)n/2]$, or $n^2/2$. The y-coordinate of P_0 is 0, and since the diagonals of a square are congruent perpendicular bisectors of each other, the y-coordinate of A_n is $n/2$, and the y-coordinate of B_n is $-n/2$. So, we are looking for an equation which defines a set of ordered pairs for which the first component is twice the square of the second. One such equation is ' $x = 2y^2$ '. [Its graph is a parabola with vertex (0, 0) and the x-axis as axis of symmetry.]

(i) 24 [24 is the only positive integral root of $\frac{n(n+1)}{2} = 300$.]

☆(j) 11 [Since $\Sigma p^3 = (\Sigma p)^2$, part (j) is like part (h), but with '721' replaced by '60'.]

5. Since $AB_1C_1 \iff AB_2C_2$ is a similarity and since $AB_2 = 2 \cdot AB_1$, it follows that $B_2C_2 = 2 \cdot B_1C_1$. Similarly, $B_3C_3 = 3 \cdot B_1C_1$, $B_4C_4 = 4 \cdot B_1C_1$, etc. So, since the altitudes to the bases of the shaded isosceles triangles are congruent, the area-measure of the second shaded region is 2 times that of the first, the area-measure of the third is 3 times that of the first, etc. Hence, the sum of the seven area-measures is

$$(1 + 2 + 3 + \dots + 7) \cdot K(\triangle AB_1C_1) = \left(\sum_{p=1}^7 p \right) \cdot 1 = \frac{7 \cdot 8}{2} = 28.$$

[A good follow-up question is to ask for the sum of the area-measures of the unshaded regions.]

6. (a) Using the formula for the area-measure of a rhombus, we find that the required sum is

$$\frac{1^2}{2} + \frac{2^2}{2} + \frac{3^2}{2} + \dots + \frac{10^2}{2} = \frac{1}{2} \sum_{p=1}^{10} p^2 = 192.5.$$

- (b) This is equivalent to asking for the sum of the perimeters of the ten squares. Since the perimeter of a square is $2\sqrt{2}$ times the measure of a diagonal, the required sum is

$$2\sqrt{2} (1 + 2 + 3 + \dots + 10) = 2\sqrt{2} \sum_{p=1}^{10} p = 110\sqrt{2}.$$

Now, $37.5 < \sqrt{1442} < 38$. So, if $\sqrt{n(n+1)} < \sqrt{1442}$, it follows, since $n < \sqrt{n(n+1)}$, that $n < 38$. On the other hand, if $n < 38$ then $n \leq 37$ and $n + \frac{1}{2} \leq 37.5$. Hence, if $n < 38$ then, since $\sqrt{n(n+1)} < n + \frac{1}{2}$, it follows that $\sqrt{n(n+1)} < 37.5 < \sqrt{1442}$. Consequently, $\sqrt{n(n+1)} < \sqrt{1442}$ if and only if $n < 38$ --that is, if and only if $n \leq 37$.]

$$(h) \ 38 \quad \left[\quad \frac{(n-1)n}{2} < 721 < \frac{n(n+1)}{2} \right]$$

$$\iff n^2 - n - 1442 < 0 \quad \text{and} \quad n^2 + n - 1442 > 0$$

$$\iff n < \frac{1 + \sqrt{5769}}{2} \quad \text{and} \quad n > \frac{-1 + \sqrt{5769}}{2}$$

$$\iff n \leq 38 \quad \text{and} \quad n \geq 38$$

$$\iff n = 38$$

In the passage from line 2 of the above to line 3 students should make use of their knowledge of quadratic functions. The first inequation of line 4 follows from that of line 3 just as in part (g) [remember that the values of 'n' are integers!]. The second components are related in a similar manner.

The alternative procedure illustrated for part (g) can also be applied here. We are looking for a positive integer n such that

$$\sqrt{(n-1)n} < \sqrt{1442} < \sqrt{n(n+1)}.$$

As in part (g) [with 'n - 1' for 'n'], the left-hand inequation is satisfied if and only if $n - 1 \leq 37$. Looking at the right-hand inequation we see that, since $n < \sqrt{n(n+1)}$ and $\sqrt{1442} < 38$, it follows that $\sqrt{n(n+1)} > \sqrt{1442}$ if $n \geq 38$. Since $\sqrt{n(n+1)} < n + \frac{1}{2}$ and $\sqrt{1442} > 37.5$, it follows that if $\sqrt{n(n+1)} > \sqrt{1442}$ then $n + \frac{1}{2} > 37.5$ --that is, that $n > 37$. So, the right-hand inequation is satisfied if and only if $n \geq 38$. Hence, both inequations are satisfied if and only if $n = 38$.]

$$\begin{aligned}
4. \text{ (g) } \{n: n \leq 37\} & \left[\frac{n(n+1)}{2} < 721 \right. \\
& \iff n(n+1) < 1442 \\
& \iff n^2 + n - 1442 < 0 \\
& \iff \left(n - \frac{-1 + \sqrt{5769}}{2} \right) \left(n - \frac{-1 - \sqrt{5769}}{2} \right) < 0 \\
& \iff n < \frac{-1 + \sqrt{5769}}{2} \\
& \iff n \leq 37
\end{aligned}$$

The final step comes from the preceding together with the fact that $75 < \sqrt{5769} < 76$ [see table on page 8-249]. From the latter it follows that

$$37 < \frac{-1 + \sqrt{5769}}{2} < 37.5.$$

Hence, if $n \leq 37$ then $n < \frac{-1 + \sqrt{5769}}{2}$; and if $n < \frac{-1 + \sqrt{5769}}{2}$ then $n < 37.5$ and, since n is an integer, $n \leq 37$.

Students should recall from their work with quadratic functions that $x^2 + x - 1442 < 0$ if and only if x is between the two roots of the equation ' $x^2 + x - 1442 = 0$ '. Since one of these roots is negative and the other positive, and since $n > 0$, $n^2 + n - 1442 < 0$ if and only if n is less than the positive root. [Make a rough sketch of the quadratic function.] There is an alternative procedure for solving exercises such as this, which involves more understanding of inequations and less computation. It is clear that $n(n+1) < 1442$ if and only if $\sqrt{n(n+1)} < \sqrt{1442}$. It should also be clear that, for even a moderately large positive integer n , $\sqrt{n(n+1)}$ is very nearly n . More precisely, since $n^2 < n(n+1) < (n + \frac{1}{2})^2$, it follows that, for each n ,

$$n < \sqrt{n(n+1)} < n + \frac{1}{2}.$$

7. [Recall that we have agreed in Unit 1 that a fraction is a numeral. Thus, the sequence in this exercise is a function whose domain is the set of positive integers and whose range is the set of all fractions whose numerators and denominators are the standard decimal numerals for the positive integers.]

One easy way to see the pattern is to add the numerator-number and the denominator-number for each fraction. The corresponding sequence of sums is

$$2, \quad | \quad 3, 3, \quad | \quad 4, 4, 4, \quad | \quad 5, 5, 5, 5, \quad | \quad 6, 6, 6, 6, 6, \quad | \quad \dots$$

So, with each fraction in the given sequence, there corresponds a sum. For each sum $s \geq 2$, a fraction for which s is the sum occurs in the $(s - 1)$ th "block". The numerator-number of the fraction tells the position of the fraction in its block.

- (a) The fraction $\frac{823}{471}$ is the 823rd term in the $(823 + 471 - 1)$ th block. There are 1292 blocks preceding this one, and so there

are a total of $\sum_{p=1}^{1292} p$ terms preceding the first term of the 1293rd

block. Hence, $\frac{823}{471}$ is the $\left(\sum_{p=1}^{1292} p + 823 \right)$ th term in the given

sequence--that is, it is the 836101st term.

- ★(b) First, we locate the block in which the 743rd term occurs. This is the $(n + 1)$ th block if and only if

$$\sum_{p=1}^n p < 743 \leq \sum_{p=1}^{n+1} p$$

--that is, if and only if

$$(*) \quad n(n + 1) < 1486 \leq (n + 1)(n + 2)$$

From the origin of $(*)$ we know that it has a unique solution--

the 743rd term is in some unique block. The unique solution, 38, of (*) can be found by either of the methods illustrated in the answer for part (h) of Exercise 4 [see COMMENTARY for page 8-21].

Here is a slight simplification of the second method, as applied to (*): From tables, $38.5 < \sqrt{1486} < 39$. Since $n < \sqrt{n(n+1)}$, it follows that to satisfy the left-hand inequation in (*) it must be the case that $n < 39$. Since $\sqrt{(n+1)(n+2)} < n + \frac{3}{2}$, it follows that to satisfy the right-hand inequation, it must be the case that $n + \frac{3}{2} > 38.5$ --that is, that $n > 37$. So, if (*) is satisfied then $37 < n < 39$. Hence, since, as noted above, (*) has a solution, the solution of (*) is 38--the 743rd term is in the 39th block.

Since the first 38 blocks contain $\sum_{p=1}^{38} p$ terms, there are 741 terms

which precede those in the 39th block, and the 743rd term is the second term of the 39th block. Hence, the 743rd term is $\frac{2}{38}$.

*

Exercise 7 shows that there is a sequence whose terms are all the fractions whose numerator-numbers and denominator-numbers are positive integers. Since each positive rational number is named by many such fractions, there also is a sequence whose terms are positive rational numbers and which is such that each positive rational number is a term [in fact, is each of many terms] of the sequence. [To describe such a sequence, replace, in the first sentence of Exercise 7, 'fraction' by 'positive rational number' and 'shown' by 'listed'.] It is not difficult to proceed from here, by omitting repetitions, to show that there is a sequence of positive rational numbers in which each positive rational number occurs exactly once. From this, we know that there are exactly as many positive rational numbers as there are positive integers.

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is no n such that $\sum_{p=1}^n p = 180$. But, if we find the largest n such that

$\sum_{p=1}^n p < 180$, we can solve the problem readily. For suppose that q is

the largest n such that $\sum_{p=1}^n p < 180$. Then, the number of cents earned is

$$\sum_{p=1}^q p(p+1) + \left(180 - \sum_{p=1}^q p\right)(q+2).$$

So, let's find q .

$$\sum_{p=1}^q p < 180 < \sum_{p=1}^{q+1} p \iff q = 18$$

Hence, the number of cents earned at the end of 3 hours is

$$\frac{18 \cdot 19 \cdot 20}{3} + \left(180 - \frac{18 \cdot 19}{2}\right)(18+2),$$

--that is, 2460. So, for 3 hours of work, the boy earned \$24.60.

*

Answer for Part D.

The completed theorem is: $\sum_{p=1}^n 1 = n$

Proof by Theorem 130: $b_1 = 1 = a_1$;

$$b_p + a_{p+1} = p + 1 = b_{p+1}$$

1. The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

where a_n are the coefficients of the power series. It is shown that the function $f(x)$ is analytic in the whole plane and that it satisfies the differential equation

$$f'(x) = f(x) + \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

It is also shown that the function $f(x)$ is the unique solution of the differential equation

$$f'(x) = f(x) + \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

which satisfies the initial condition $f(0) = 1$.

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

$$(d) \sum_{p=1}^{100} p(p+1)(p+2)(p+3) = \frac{100 \cdot 101 \cdot 102 \cdot 103 \cdot 104}{5} = 2207100480$$

(e) The required sum is 8 times the sum in (c). So, the required sum is 212221200.

| | | |
|----|---|--------------------------|
| 4. | <u>The boy has earned</u> | <u>at the end of</u> |
| | $1 \cdot 2$ cents | 1 minutes |
| | $1 \cdot 2 + 2 \cdot 3$ cents | $1 + 2$ minutes |
| | $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4$ cents | $1 + 2 + 3$ minutes |
| | \vdots | \vdots |
| | $\sum_{p=1}^n p(p+1)$ cents | $\sum_{p=1}^n p$ minutes |
| | \vdots | \vdots |

Now, suppose that the boy has worked for m minutes. If there is an

n such that $\sum_{p=1}^n p = m$, it is easy to find the amount earned at the end

of m minutes. This is the case in the first problem.

$$\sum_{p=1}^n p = 120 \iff \frac{n(n+1)}{2} = 120 \iff n = 15$$

$$\sum_{p=1}^{15} p(p+1) = \frac{15 \cdot 16 \cdot 17}{3} = 1360$$

So, for two hours of work, the boy earned \$13.60.

★The second problem is more challenging. For in this case, there

Answers for Part C.

$$1. (a) \frac{q(q+1)(q+2)}{3} + (q+1)(q+2) = \frac{q(q+1)(q+1) + 3(q+1)(q+2)}{3} \\ = \frac{(q+1)(q+2)(q+3)}{3}$$

(b) Theorem 130 and (**) make this task almost trivial. For, if $a_p = p(p+1)$ then (**) tells us that, for $b_q = \frac{q(q+1)(q+2)}{3}$, $b_q + a_{q+1} = b_{q+1}$. So, we have only to check that $a_1 = 1 \cdot 2 = \frac{1 \cdot 2 \cdot 3}{3} = b_1$ to discover and prove [by Theorem 130] that

$$\forall_n \sum_{p=1}^n p(p+1) = \frac{n(n+1)(n+2)}{3}.$$

$$2. (a) \frac{q(q+1)(q+2)(q+3)}{4} + (q+1)(q+2)(q+3) \\ = \frac{q(q+1)(q+2)(q+3) + 4(q+1)(q+2)(q+3)}{4} \\ = \frac{(q+1)(q+2)(q+3)(q+4)}{4}$$

$$(b) \forall_n \sum_{p=1}^n p(p+1)(p+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

The proof is easy since the manipulation is embodied in the algebra theorem of part (a).

$$3. (a) \sum_{p=1}^{100} p = \frac{100 \cdot 101}{2} = 5050 \quad (b) \sum_{p=1}^{100} p(p+1) = \frac{100 \cdot 101 \cdot 102}{3} = 343400$$

$$(c) \sum_{p=1}^{100} p(p+1)(p+2) = \frac{100 \cdot 101 \cdot 102 \cdot 103}{4} = 26527650$$

Ask students to state the theorems which would be parts e and f of Theorem 132. Later, students may prove a generalization [Theorem 179b], whose instances are equivalent to the parts of Theorem 132.

Later parts of Theorem 131 are not easy to predict. [It can, of course, be done. See, for example Hall and Knight's Higher Algebra [London: Macmillan, 1948], pages 336 and 337.] For your amusement, here is a list of formulas for sums of powers:

$$S_1(n) = \frac{n(n+1)}{2} = m, \text{ say,}$$

$$S_2(n) = (2n+1)\frac{m}{3},$$

$$S_3(n) = m^2,$$

$$S_4(n) = (2n+1)\left(\frac{2}{5}m^2 - \frac{1}{15}m\right),$$

$$S_5(n) = \frac{4}{3}m^3 - \frac{1}{3}m^2,$$

$$S_6(n) = (2n+1)\left(\frac{4}{7}m^3 - \frac{2}{7}m^2 + \frac{1}{21}m\right),$$

$$S_7(n) = 2m^4 - \frac{4}{3}m^3 + \frac{1}{3}m^2,$$

$$S_8(n) = (2n+1)\left(\frac{8}{9}m^4 - \frac{8}{9}m^3 + \frac{2}{5}m^2 - \frac{1}{15}m\right),$$

$$S_9(n) = \frac{16}{5}m^5 - 4m^4 + \frac{12}{5}m^3 - \frac{3}{5}m^2,$$

$$S_{10}(n) = (2n+1)\left(\frac{16}{11}m^5 - \frac{80}{33}m^4 + \frac{68}{33}m^3 - \frac{10}{11}m^2 + \frac{5}{33}m\right)$$

[Note that $\forall_n S_5(n) + S_7(n) = 2[S_3(n)]^2$. This is only moderately difficult to prove by induction.]

*

Part ☆E, which begins on page 8-24, is discussed in the COMMENTARY for page 8-25.

5. There are at least two such theorems derivable from the theorem of Exercise 2. From the latter it follows that, for each n ,

$$\sum_{p=1}^n f_p + \sum_{p=1}^{n+1} f_p = f_{n+2} - 1 + f_{n+3} - 1 = f_{n+4} - 2 \quad [\text{first theorem}],$$

$$= \sum_{p=1}^{n+2} f_p - 1 \quad [\text{second theorem}].$$

6. Suppose that the initial pair of adult rabbits is procured on Jan. 1 and that births occur on the last day of each month. Tabulating the results expected we have:

| | Jan.1 | Jan.31 | Feb.28 | Mar.31 | | Nov.30 | Dec.31 |
|------------------------------|-------|--------|--------|--------|--|----------|----------|
| Pairs of adults | 1 | 1 | 2 | 3 | | f_{12} | f_{13} |
| Pairs produced on given date | | 1 | 1 | 2 | | f_{11} | f_{12} |

The horizontal and the upward-slanting arrows indicate the sources of adult pairs, and the downward-slanting arrows indicate genealogical descent. The second line makes clear that the total number

of pairs produced is $\sum_{p=1}^{12} f_p$. By the theorem of Exercise 2, this is

$f_{14} - 1$ --that is, 376. [Alternatively, the total number of pairs produced is 1 less than the sum of the numbers of adult and nonadult pairs on Dec. 31. This is $f_{13} + f_{12} - 1$, and, by the recursive definition, again, $f_{14} - 1$.]

Part ★E is the first of several sets of optional exercises dealing with the Fibonacci sequence f:

$$\begin{cases} f_1 = 1 \\ f_2 = 1 \\ \forall_n f_{n+2} = f_n + f_{n+1} \end{cases}$$

and its generalizations. [These parts are referred to in the COMMENTARY for page 8-20.] Make sure, in connection, say, with solving Exercise 1, that students see that the sum of any two successive terms is the next term--that is, don't substitute in ' $f_{n+2} = f_n + f_{n+1}$ ', but, instead, "do what it says".

Answers for Part ★E.

1. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55

2. 1, 2, 4, 7, 12, 20, 33, 54; the theorem is: $\forall_n \sum_{p=1}^n f_p = f_{n+2} - 1$

3. Suppose that, for each n, $a_n = f_n$ and $b_n = f_{n+2} - 1$. Then,

$$a_1 = f_1 = 1 = 2 - 1 = f_{1+2} - 1 = b_1. \text{ Also, for each n,}$$

$$b_n + a_{n+1} = f_{n+2} - 1 + f_{n+1} = f_{n+3} - 1 = b_{n+1}. \text{ So, by}$$

$$\text{Theorem 130, } \forall_n \sum_{p=1}^n a_p = b_n \text{--that is, } \forall_n \sum_{p=1}^n f_p = f_{n+2} - 1.$$

4. The odd-numbered terms of f are 1, 2, 5, 13, 34, ..., and the continued sum sequence for this sequence is 1, 3, 8, 21, 55, These seem to be the even-numbered terms of f, suggesting that

$$\forall_n \sum_{p=1}^n f_{2p-1} = f_{2n}.$$

This is easy to prove, using Theorem 130: $f_{2 \cdot 1} - 1 = 1 = f_{2 \cdot 1}$;
 $f_{2p} + f_{2p+1} = f_{2p+2} = f_{2(p+1)}.$

Answers for Miscellaneous Exercises.

| | |
|----------------|--|
| <u>Easy:</u> | A1-12; B1-5; C1, 3, 4, 5d-l, 7a, 10, 12-30, 32, 34 |
| <u>Medium:</u> | A13; C2, 6 (tricky), 7b, 9, 33 |
| <u>Hard:</u> | B6-9; C5a-c, 8, 11 (long), 31 |

Students should discover, as illustrated in the solution for the Sample in Part A, that in order to find which corresponding terms of sequences a and b are equal, all that one does is to find all positive integers p such that $a_p = b_p$. In the case at hand, this involves solving the equation ' $3p + 4 = 2p + 9$ '. Since this equation has just the root 5, only the fifth terms are corresponding equal terms.

The more general question as to which terms of a are also terms of b leads to the Diophantine problem of solving ' $3p + 4 = 2q + 9$ '. Students who have learned in Unit 7 how to solve such problems may like to practice their skill on similar modifications of the first 8 exercises of Part A. [The remaining exercises, when so modified, lead to second degree Diophantine problems, and will prove quite difficult.] For the Sample, one sees that the solutions of ' $3p - 2q = 5$ ' are the pairs $(1+2k, -1+3k)$, for $k \geq 1$. So, for each n , $a_{2n+1} = b_{3n-1}$. From this, incidentally, it is easy to see that the only pair of corresponding equal terms is (a_5, b_5) [$2n + 1 = 3n - 1$ if and only if $n = 2$].

You might ask students if they could find the corresponding terms by graphing the sequences. Try to get students to predict the outcome of a graphing approach ["Each graph is a set of dots, and the sets intersect at one point. The first component of this point is the required term number."]. Then, actually carry out the graphing. Graphical interpretations are instructive for all the exercises in this part, but particularly for Exercises 3, 4, 6, 8, 9, 10, and 11.

*

Answers for Part A.

1. $a_{10} = b_{10}$
2. $a_2 = b_2$
3. no terms $[\{p: 5p + 4 = 3p + 7\} = \emptyset]$
4. no terms
5. $a_7 = b_7$
6. $a_4 = b_4$
7. $a_4 = b_4$
8. all corresponding terms [That is, $a = b$. We have violated the exercise description by not giving two sequences in this exercise.]
9. $a_7 = b_7$ and $a_{12} = b_{12}$
10. no terms
11. $a_2 = b_2$
12. no terms
13. $a_8 = b_8$

9. By the algebra theorem proved in Exercise 8,

$$\frac{q(q+1)(q+2\cdot\frac{k}{2})(q+2\cdot\frac{k}{2}+1)}{4} + (q+1)(q+1+\frac{k}{2})(q+1+2\cdot\frac{k}{2})$$

$$= \frac{(q+1)(q+1+1)(q+1+2\cdot\frac{k}{2})(q+1+2\cdot\frac{k}{2}+1)}{4}.$$

$$\text{So, } \frac{q(q+1)(q+k)(q+k+1)}{4} + \frac{(q+1)[2(q+1)+k][q+1+k]}{2}$$

$$= \frac{(q+1)(q+1+1)(q+1+k)(q+1+k+1)}{4},$$

$$\text{--that is, } \frac{q(q+1)(q+k)(q+k+1)}{2} + (q+1)(q+1+k)[2(q+1)+k]$$

$$= \frac{(q+1)(q+1+1)(q+1+k)(q+1+k+1)}{2}.$$

This result leads immediately to the algebra theorem we need in establishing the generalization in Exercise 9. [Of course, with Theorem 133 available, we could get Exercise 9 right from Exercise 8 just by replacing the 'k's by 'k/2's and doing a bit of manipulation.]

*

Students may enjoy checking the theorems in Part B against one another and against earlier theorems. It is also very worthwhile to develop the habit of checking a result [or a guess] against one's previous knowledge. This not only helps to guard against errors, but also develops one's power of guessing a general result on the basis of more special ones. For example, the instance of Exercise 8 for $k = 1$ should check with Theorem 132d, and the instance for $k = 0$ should check with Theorem 131d. Also, the instance of Exercise 9 for $k = 1$ should check with Exercise 5, and the instance of Exercise 7 for $k = 2$ should check with Exercise 4. Exercise 7 can also be checked against parts of Theorems 131 and 132 and, perhaps, by your better students, against Exercise 1 [in doing so they may discover Theorem 137]. It also checks with Exercise 3. Exercise 2 can be checked against Theorem 132d. Finally, comparing Exercises 8 and 9, students may see [anticipating Theorem 133] that the first is a special case of the second and that, in either one [as well as in Exercise 7], 'k' may be replaced throughout by 'x'.

$$7. \frac{q(q+1)(2q+1+3k)}{6} + (q+1)(q+1+k)$$

$$= \frac{(q+1)(2q^2 + q + 3kq + 6q + 6 + 6k)}{6}$$

$$= \frac{(q+1)[(2q^2 + 7q + 6) + (3kq + 6k)]}{6}$$

$$= \frac{(q+1)[(q+2)(2q+3) + 3k(q+2)]}{6}$$

$$= \frac{(q+1)(q+2)(2q+3+3k)}{6}$$

$$= \frac{(q+1)(q+1+1)[2(q+1)+1+3k]}{6}$$

$$8. \frac{q(q+1)(q+2k)(q+2k+1)}{4} + (q+1)(q+1+k)(q+1+2k)$$

$$= \frac{(q+1)(q+2k+1)[q(q+2k) + 4(q+1+k)]}{4}$$

$$= \frac{(q+1)(q+2k+1)(q^2 + 2kq + 4q + 4 + 4k)}{4}$$

$$= \frac{(q+1)(q+2k+1)[(q+2)^2 + 2k(q+2)]}{4}$$

$$= \frac{(q+1)(q+2)(q+2k+1)(q+2+2k)}{4}$$

$$= \frac{(q+1)(q+1+1)(q+1+2k)(q+1+2k+1)}{4}$$

$$\begin{aligned}
 4. \quad \frac{q(q+1)(2q+7)}{6} + (q+1)(q+3) &= \frac{(q+1)[q(2q+7) + 6(q+3)]}{6} \\
 &= \frac{(q+1)(2q^2 + 13q + 18)}{6} \\
 &= \frac{(q+1)(q+2)(2q+9)}{6} \\
 &= \frac{(q+1)(q+1+1)[2(q+1)+7]}{6}
 \end{aligned}$$

$$\begin{aligned}
 5. \quad \frac{q(q+1)^2(q+2)}{2} + (q+1)(q+2)(2q+3) &= \frac{(q+1)(q+2)[q(q+1) + 2(2q+3)]}{2} \\
 &= \frac{(q+1)(q+2)(q^2 + 5q + 6)}{2} \\
 &= \frac{(q+1)(q+1+1)^2(q+1+2)}{2}
 \end{aligned}$$

6. [The algebra in this problem is probably easier if we proceed in the reverse direction.]

$$\begin{aligned}
 \frac{(q+1)[6(q+1)^2 + 15(q+1) + 11]}{2} &= \frac{(q+1)[(6q^2 + 15q + 11) + (12q + 21)]}{2} \\
 &= \frac{q(6q^2 + 15q + 11) + 1(6q^2 + 15q + 11) + (q+1)(12q + 21)}{2} \\
 &= \frac{q(6q^2 + 15q + 11) + 2(9q^2 + 24q + 16)}{2} \\
 &= \frac{q(6q^2 + 15q + 11)}{2} + (3q + 4)^2 \\
 &= \frac{q(6q^2 + 15q + 11)}{2} + [3(q+1) + 1]^2
 \end{aligned}$$

Answers for Part B.

$$\begin{aligned} 1. \quad \frac{q(q+4)(q+5)}{3} + (q+2)(q+5) &= \frac{(q+5)[q(q+4) + 3(q+2)]}{3} \\ &= \frac{(q+5)(q^2 + 7q + 6)}{3} \\ &= \frac{(q+1)(q+1+4)(q+1+5)}{3} \end{aligned}$$

$$\begin{aligned} 2. \quad \frac{q(q^2 - 1)(q + 2)}{4} + [(q + 1)^3 - (q + 1)] &= \frac{q(q^2 - 1)(q + 2) + 4(q + 1)[(q + 1)^2 - 1]}{4} \\ &= \frac{(q^2 - 1)(q^2 + 2q) + 4(q + 1)[(q + 1)^2 - 1]}{4} \\ &= \frac{(q^2 - 1)[(q + 1)^2 - 1] + 4(q + 1)[(q + 1)^2 - 1]}{4} \\ &= \frac{(q + 1)[(q + 1)^2 - 1](q - 1 + 4)}{4} \\ &= \frac{(q + 1)[(q + 1)^2 - 1](q + 1 + 2)}{4} \end{aligned}$$

$$\begin{aligned} 3. \quad \frac{q(q-1)(2q+5)}{6} + [(q+1)^2 - 1] &= \frac{q[(q-1)(2q+5) + 6(q+2)]}{6} \\ &= \frac{q(2q^2 + 9q + 7)}{6} \\ &= \frac{q(q+1)(2q+7)}{6} \\ &= \frac{(q+1)(q+1-1)[2(q+1)+5]}{6} \end{aligned}$$

Answers for Exercises 12 and 13 of Part A are given in the COMMENTARY for page 8-26.

*

Mathematical induction or Theorem 130 may be used for the proofs of Part B. Some students may invent methods which consist of applying Theorems 131 and 132. But, in doing so, they will be anticipating Theorems 133 and 134. They should realize this, and thus become aware of a need for proving Theorems 133 and 134. For example, consider Exercise 1.

$$\begin{aligned}
 \sum_{p=1}^q (p+1)(p+4) &= \sum_{p=1}^q (p^2 + 5p + 4) \\
 &= \sum_{p=1}^q p^2 + \sum_{p=1}^q 5p + \sum_{p=1}^q 4 \cdot 1 \\
 &= \sum_{p=1}^q p^2 + 5 \sum_{p=1}^q p + 4 \sum_{p=1}^q 1 \\
 &= \frac{q(q+1)(2q+1)}{6} + \frac{5q(q+1)}{2} + 4q \\
 &= \frac{q(q+1)(2q+1) + 15q(q+1) + 24q}{6} \\
 &= \frac{q(2q^2 + 3q + 1 + 15q + 15 + 24)}{6} \\
 &= \frac{q(q^2 + 9q + 20)}{3} = \frac{q(q+4)(q+5)}{3}
 \end{aligned}
 \begin{array}{l}
 \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{Th. 134} \\
 \left. \begin{array}{l} \\ \end{array} \right\} \text{Th. 133} \\
 \left. \begin{array}{l} \\ \end{array} \right\} \text{Th. 131}
 \end{array}$$

In the following answers we give only the algebra needed for application of Theorem 130, or proof by induction.

*

Answers for Part C.

1. \$400 at 5% and \$600 at $3\frac{1}{2}\%$

2. (a) $\frac{4j^2 - 7j}{(j+1)(j+2)(j-1)}$

(b) $-\frac{8q^2 + 25q + 28}{(q+2)(q^2 + 3q + 4)}$

3. (a) No real roots.

(b) $-\frac{1}{2}$

4. 135

5. (a) $(y-1-x)(y-1+x)$

(b) $(t+u-s)(t+u+s)$

(c) $k(k-2)[(k-1)^2 + 1]$

(d) $(x-8)(x-3)$

(e) $3(x-3)(x+3)$

(f) $\frac{1}{4}x(x-1)(x+1)$

(g) $4(t+2)^2$

(h) $3(a+14)^2$

(i) $(3+y)(2-y)$

(j) $(a-5b)(a-12b)$

(k) $(x+10y)(x-9y)$

(l) $(2x-3)(x+2)$

6. 32

7. (a) -2.5

(b) 2

8. This exercise in "partial fractions" should prove to be a real challenge since students must invent their own techniques for solving it. The job is to find real numbers a , b , and c such that for each $x \neq 1$,

$$\frac{2x^2 - 3x - 5}{(x^2 + 2)(x - 1)} = \frac{ax + b}{x^2 + 2} + \frac{c}{x - 1} = \frac{(ax + b)(x - 1) + c(x^2 + 2)}{(x^2 + 2)(x - 1)}.$$

Clearly, this will be the case if, for each x ,

$$(*) \quad 2x^2 - 3x - 5 = (ax + b)(x - 1) + c(x^2 + 2).$$

We try to solve this simpler problem. If $(*)$ is to be satisfied by each real number, it must be satisfied by the number 1. So,

$$2 \cdot 1^2 - 3 \cdot 1 - 5 = (a \cdot 1 + b)(1 - 1) + c(1^2 + 2)$$

--that is, $3c = -6$, and $c = -2$. With this choice for c , (*) is equivalent to:

$$(**) \quad 4x^2 - 3x - 1 = (ax + b)(x - 1)$$

and, since, for each x ,

$$4x^2 - 3x - 1 = (4x + 1)(x - 1)$$

(**) will be satisfied for all x if $ax + b = 4x + 1$. This suggests that

$$\forall x \neq 1 \quad \frac{2x^2 - 3x - 5}{(x^2 + 2)(x - 1)} = \frac{4x + 1}{x^2 + 2} + \frac{-2}{x - 1}.$$

The preceding discussion does no more than make it seem likely that this is the case. Completion of this exercise requires checking that this is so.

Students will probably discover other ways of guessing the solution [needless to say, this activity should be encouraged]. In any case, their only recourse, after guessing, will be an algebraic check that their guesses are correct.

There is a theorem according to which there does exist a unique solution (a, b, c) for the problem of this exercise, and a theorem according to which (a, b, c) is a solution for the problem, not only if, but also only if it is a solution to the problem set by (*). From the preceding discussion and the second of these theorems it follows that if the given problem has a solution, this solution is $(4, 1, -2)$. From this and the first of the two theorems referred to it follows that $(4, 1, -2)$ is a solution--and there is no need for a further check that this is so. However, neither theorem [let alone proofs of the theorems] is accessible to students at this point. The first requires the concept of prime factors of a polynomial expression and the second requires the concept of continuity. [See any calculus text or Hall and Knight, for an exposition of the partial fraction technique. For justification of the technique see, for example, N. H. McCoy's Introduction to Modern Algebra [Boston: Allyn and Bacon, 1960].]

9. (a) $\frac{-b^2}{a(a-b)^2}$

(b) $\frac{9t^2 - 5tu + u^2}{t - u}$

1900

1901

1902

1903

1904

1905

1906

1907

1908

1909

1910

1911

1912

1913

1914

1915

1916

23. (a) $-7(m + n)$

(b) $(a - b)(a^2 + b^2)$

(c) $15a^4/14$

24. $16t$

25. 6 inches

26. (a) $12x - 28y - 10xy$

(b) $2(a + b) + 3a^2b^2$

(c) $6(s - t)$

(d) $4a^2b - 6ab^2 + 1$

27. (a) $3/4$

(b) $(x - y)/(x + y)^2$

28. (a) 7; 3

(b) $2/7; -5/7$

(c) $7/3; -2$

(d) 3; -7

(e) 2.1; -3.5

(f) $3/4; -19/4$

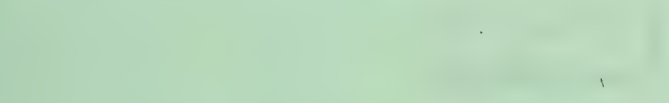
1. The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \int_0^x f(t) dt + g(x)$$

where $g(x)$ is a given function. It is shown that if $g(x)$ is continuous and has a bounded derivative, then $f(x)$ is also continuous and has a bounded derivative. The proof is based on the fact that the integral of a bounded function is continuous and has a bounded derivative.

2. In the second part of the paper, we consider the problem of finding the maximum value of the function $f(x)$ on the interval $[a, b]$. It is shown that if $f(x)$ is continuous and has a bounded derivative, then the maximum value of $f(x)$ on $[a, b]$ is attained at one of the endpoints of the interval.

3. The third part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt + g(x)$ where $g(x)$ is a given function. It is shown that if $g(x)$ is continuous and has a bounded derivative, then $f(x)$ is also continuous and has a bounded derivative. The proof is based on the fact that the integral of a bounded function is continuous and has a bounded derivative.



4. The fourth part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt + g(x)$ where $g(x)$ is a given function. It is shown that if $g(x)$ is continuous and has a bounded derivative, then $f(x)$ is also continuous and has a bounded derivative. The proof is based on the fact that the integral of a bounded function is continuous and has a bounded derivative.

5. The fifth part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt + g(x)$ where $g(x)$ is a given function. It is shown that if $g(x)$ is continuous and has a bounded derivative, then $f(x)$ is also continuous and has a bounded derivative. The proof is based on the fact that the integral of a bounded function is continuous and has a bounded derivative.

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8. The eighth part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt + g(x)$ where $g(x)$ is a given function. It is shown that if $g(x)$ is continuous and has a bounded derivative, then $f(x)$ is also continuous and has a bounded derivative. The proof is based on the fact that the integral of a bounded function is continuous and has a bounded derivative.

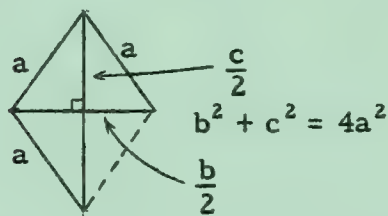
9. The ninth part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt + g(x)$ where $g(x)$ is a given function. It is shown that if $g(x)$ is continuous and has a bounded derivative, then $f(x)$ is also continuous and has a bounded derivative. The proof is based on the fact that the integral of a bounded function is continuous and has a bounded derivative.

14. This is a surprising result. It can be established with the help of the area-measure formula:

$$K = \sqrt{s(s-a)(s-b)(s-c)},$$

where a , b , and c are the side-measures and s is the semiperimeter. Or, one can compute the measure of the altitude to the base of each of these isosceles triangles. For the smaller base, the altitude's measure is 40, and for the larger, it is 30. So, for each triangle, the area-measure is 1200.

An insight into the result, and an overview of all such results, can be obtained by noticing that the area-measure of each triangle is half the area-measure of a rhombus of side-measure 50 whose diagonals have measures 60 and 80.



In general, (a, a, b) and (a, a, c) are the triples of side-measures of pairs of isosceles triangles with the same area-measure just if b and c are the measures of the diagonals of a rhombus whose side-measure is a . That is, just if $b^2 + c^2 = 4a^2$. A quick way

to find numbers b and c , given a , is to use the parametric formulas:

$$\begin{cases} b = 2a \cos \theta \\ c = 2a \sin \theta \end{cases}$$

15. (a) $(5, -2)$ [It is easier not to clear the equations of fractions.]

(b) $(-3, 15)$

16. 20

17. (a) 0.6

(b) $\frac{25}{4}$

(c) $\frac{5}{53}$

18. (a) 7200

(b) $\frac{1}{2}$

(c) 6

19. No

20. $6c$

21. $\frac{25x^2}{4}$

22. (a) $-5x^2$

(b) $6x + 10y$

(c) $2y^2 - 3y + 2$

10. $r/(r - 10)$

11. (a) -1 and $\frac{1}{2}$ (b) Yes (c) Yes (d) No (e) No

(f) [Principally] by Theorem 99a, it follows that, for $-1 \neq x \neq \frac{1}{2}$,

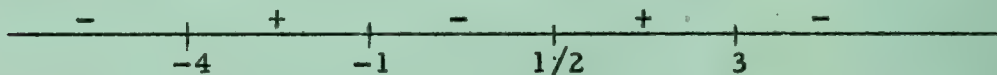
$$\frac{(3 - x)(x + 4)}{(2x - 1)(x + 1)} > 0 \iff (3 - x)(x + 4)(2x - 1)(x + 1) > 0.$$

At this point the ftpi [specifically, Theorem 96b] is applicable. A systematic procedure for applying this theorem consists in making a table:

| | $x + 4$ | $x + 1$ | $2x - 1$ | $3 - x$ | Product |
|------------------------|---------|---------|----------|---------|---------|
| $x < -4$ | - | - | - | + | - |
| $-4 < x < -1$ | + | - | - | + | + |
| $-1 < x < \frac{1}{2}$ | + | + | - | + | - |
| $\frac{1}{2} < x < 3$ | + | + | + | + | + |
| $3 < x$ | + | + | + | - | - |

So, $f(x) > 0 \iff (-4 < x < -1 \text{ or } 1/2 < x < 3).$

After sufficient practice on this sort of thing one replaces the table by:



which is obtained by checking that, for $x < -4$, an odd number of factors have negative values, writing, therefore, a '-' to the left of the '-4', and then alternating '+'s and '-'s. But, before going on to this stage, one should solve similar problems involving such factors as $(x - 1)^2$ and $(x + 2)^3$.

12. 1 and 2 [But, $x^2 + 4x - 21 < 0 \iff -7 < x < 3.$]

13. $-10x^2 - x + 1$ [Ask students to complete when the '+' before the '(' is replaced by a '-'.]

$$2. \quad \sum_{p=1}^q c = \sum_{p=1}^q c \cdot 1 = c \sum_{p=1}^q 1 = c \cdot q.$$

\uparrow pm 1 \uparrow Ex. 1 \uparrow Th. 131

Consequently [two applications of the test pattern principle],

$$\forall_x \forall_n \sum_{p=1}^n x = x \cdot n.$$

$$3. \quad (a) \quad \sum_{p=1}^{50} p = 1275 \qquad (b) \quad \sum_{p=1}^{50} 7p = 7 \sum_{p=1}^{50} p = 7 \cdot 1275 = 8925$$

$$(c) \quad \sum_{p=1}^{50} \frac{p}{3} = \frac{1}{3} \sum_{p=1}^{50} p = \frac{1}{3} \cdot 1275 = 425$$

$$(d) \quad \sum_{p=1}^{50} \frac{p}{50} = \frac{1}{50} \sum_{p=1}^{50} p = \frac{1}{50} \cdot 1275 = \frac{51}{2}$$

$$(e) \quad \sum_{p=1}^{50} 4 = 4 \cdot 50 = 200$$

33. For this problem we must find numbers $a \neq 0$, b , and c such that

$$\{(1, -3), (3, 5), (-1, 5)\} \subseteq \{(x, y): y = ax^2 + bx + c\}.$$

And, to do this amounts to solving the system of equations:

$$\begin{cases} a + b + c = -3 \\ 9a + 3b + c = 5 \\ a - b + c = 5 \end{cases}$$

The sought-for quadratic function is $\{(x, y): y = 2x^2 - 4x - 1\}$.

[Much labor is saved if one notices that two of the points-- $(3, 5)$ and $(-1, 5)$ --are symmetric with respect to $\{(x, y): x = 1\}$. So, this is the axis of symmetry, and since the third point-- $(1, -3)$ --belongs to the axis of symmetry, it is the vertex. Hence, all that remains is to find a number $a \neq 0$ such that, say, $(3, 5)$ belongs to $\{(x, y): y = a(x - 1)^2 - 3\}$.]

34. (a) no roots

(b) -2 [Note that 3 is not a root.]

*

Answers for Exploration Exercises.

1. By Theorem 130, since $ca_1 = c \sum_{p=1}^1 a_p$ and

$$c \sum_{p=1}^q a_p + ca_{q+1} = c \left[\sum_{p=1}^q a_p + a_{q+1} \right] = c \sum_{p=1}^{q+1} a_p,$$

it follows that

$$\forall_n \sum_{p=1}^n ca_p = c \sum_{p=1}^n a_p.$$

Consequently [test-pattern principle], $\forall_x \forall_n \sum_{p=1}^n xa_p = x \sum_{p=1}^n a_p$.

[Alternatively, the theorem can be proved by mathematical induction. This involves little more than a rearrangement of the work done in the proof given above.]

[illegible]

29. $\frac{12x^2 - 45x + 46}{4}$

30. (a) $2\sqrt{5}$ (b) $7\sqrt{3} - 3\sqrt{5}$ (c) 7 (d) 16 (e) $106 - 58\sqrt{3}$

31. Suppose that the farmer has p feet of fencing and wishes to build k pens with the two east-west fences each x feet long. To do so, he must use $(p - 2x)/(k + 1)$ feet of fencing for each of the $k + 1$ north-south fences. So, the measure of the total region enclosed is $x(p - 2x)/(k + 1)$. Since k is given and $k + 1 > 0$, the problem of maximizing the area-measure reduces to that of maximizing $x(p - 2x)$. Since

$$x(p - 2x) = -2(x^2 - \frac{p}{2}x) = -2[(x - \frac{p}{4})^2 - \frac{p^2}{16}],$$

the maximum is attained when $x = p/4$ --that is, when half the fencing is used for the east-west fences [and the other half for the north-south fences].

It is somewhat surprising that the result is independent of the number of pens. You might ask students to guess, before doing this exercise, whether more fencing will be used for north-south fences in, say, the case of 3 pens than in the case of 2 pens. After the exercise is solved you might ask for the total area-measure

$$\frac{\frac{p}{4}(p - 2 \cdot \frac{p}{4})}{k + 1} = \frac{p^2}{8(k + 1)}$$

and note that it is directly proportional to the square of the measure of available fencing and is inversely proportional to the number of pens increased by 1.

32. $x = \frac{-q \pm \sqrt{q^2 - 4pr}}{2p}$ [$p \neq 0$, $q^2 \geq 4pr$]

[Encourage students to do this by completing the square.]

Be sure students realize that, for any $q \in I^+$ and any real number c ,

$\sum_{p=1}^q c$ is the sum of the first q terms of the constant sequence each of

whose terms is c .

*

One consequence of (*) is:

$$(1) \quad \forall_x \sum_{p=1}^1 x = x \cdot 1$$

Since, by the recursive definition of Σ -notation [page 8-9],

$$\sum_{p=1}^1 x = x,$$

it follows from (1) that

$$(2) \quad \forall_x x = x \cdot 1.$$

So, [by symmetry of equality],

$$(3) \quad \forall_x x \cdot 1 = x.$$

... ..

 (100)

... ..

 (100)

... .. (1)

... ..

1. The first part of the report is a general introduction to the subject of the study. It discusses the importance of the study and the objectives of the research.

2. The second part of the report is a detailed description of the methodology used in the study. It includes information about the sample, the data collection methods, and the statistical analysis.

3. The third part of the report is a discussion of the results of the study. It presents the findings of the research and discusses their implications for the field of study. It also includes a comparison of the results with previous research.

4. The fourth part of the report is a conclusion and a list of references. The conclusion summarizes the main findings of the study and provides recommendations for future research. The references list the sources of information used in the study.

5. The fifth part of the report is a list of appendices. These include any additional information that is relevant to the study, such as questionnaires, interview schedules, and raw data.

6. The sixth part of the report is a list of figures and tables. These provide a visual representation of the data and are used to illustrate the results of the study.

7. The seventh part of the report is a list of footnotes. These provide additional information about the study, such as the author's contact information and any acknowledgments.

8. The eighth part of the report is a list of abbreviations. These are used to simplify the text and make it easier to read.

9. The ninth part of the report is a list of symbols. These are used to represent mathematical concepts and are explained in the text.

10. The tenth part of the report is a list of definitions. These are used to clarify the meaning of key terms and concepts used in the study.

11. The eleventh part of the report is a list of acknowledgments. These are used to thank the people and organizations that have supported the study.

One advantage of the more general definition (††) of Σ -notation is that it justifies writing, say:

$$\sum_{p=1}^q p^2 = \sum_{p=1}^{q-1} p^2 + q^2$$

without the restriction ' $[q \geq 2]$ ' which would be necessitated if one used the less general definition (†). This freedom simplifies some proofs which would otherwise have to include special treatment of, say, the case $q = 1$. [An example like this occurs in passing from the 3rd to 4th lines of the proof of (*) on page 8-52. The only difference is that, there, '2' replaces '1' and ' $q + 1$ ' replaces ' q '.]

[The cause of avoiding meaningless expressions would be better served by replacing the first sentence of (††) by:

$$\forall_{k \leq 0} \sum_{p=1}^k a_p = 0,$$

and, although we do not do it, this is often done.]

One can think of $\sum_{p=1}^0 a_p$ as the "sum of no terms" of the sequence a .

It is clear that we need to make some kind of fuss about extending the meaning of 'sum' this way.

In (††) we use ' k ' instead of ' n ' because we now need a variable whose domain includes 0 as well as the positive integers. [But the domain of ' k ', by an earlier convention, includes the negative integers, also. Hence, the restriction on the quantifier in the second sentence of (††).]

The consequence of (**) which does not follow from (*) is ' $\forall_x x \cdot 0 = 0$ ' which tells us that "multiplying by 0 is repeated addition where we don't even get started adding".

In the fourth line from the bottom the sentence 'for each n , $n \geq 0$ ' tells us, in effect, that the domain of ' n ' is a subset of the domain of ' k '. So, each instance of the second sentence of (†) is also an instance of the second sentence of (††).

Answers for Part A.

1. 0 2. 0 3. 0 [The fraction is a "red herring".]
 4. 1 5. 0 6. 0 7. $5 \cdot 7$ [or: 35] 8. 0 9. 0

*

Answers for Part B.

1. $g_0 = 3 \cdot 0 + 7 = 7$; $g_1 = 10$; $g_2 = 13$; $g_7 = 28$
 [Notice that the domain of g is not \mathbb{I}^+ . That is why we don't refer to g as a sequence.]
2. $a_{-7} = -13$; $a_{-4} = -7$; $a_0 = 1$; $a_{-1} = -1$
3. $-13 + -11 + -9 + -7 + -5$ [or: -45]
4. We say 'guess' because the symbol has not yet been defined. We hope that Exercise 3 has prepared the way for the "right" guess:

$$\cdot \sum_{k=-7}^{-3} (2k + 1) \text{ means } -13 + -11 + -9 + -7 + -5$$

$$5. \cdot \sum_{p=3}^7 [2(p - 10) + 1] \text{ means } -13 + -11 + -9 + -7 + -5$$

*

Answers for Part C.

1. Hope for: $\sum_{p=3}^5 (7p + 2)$ but accept: $\sum_{p=1}^5 (7p + 2) - \sum_{p=1}^2 (7p + 2)$

2. $\sum_{p=4}^5 p^2$

3. $\sum_{p=6}^9 2p$

4. $\sum_{p=4}^{11} p^2$

1. 1941 1942 1943 1944 1945 1946 1947 1948 1949 1950 1951 1952 1953 1954 1955 1956 1957 1958 1959 1960 1961 1962 1963 1964 1965 1966 1967 1968 1969 1970 1971 1972 1973 1974 1975 1976 1977 1978 1979 1980 1981 1982 1983 1984 1985 1986 1987 1988 1989 1990 1991 1992 1993 1994 1995 1996 1997 1998 1999 2000 2001 2002 2003 2004 2005 2006 2007 2008 2009 2010 2011 2012 2013 2014 2015 2016 2017 2018 2019 2020 2021 2022 2023 2024 2025 2026 2027 2028 2029 2030 2031 2032 2033 2034 2035 2036 2037 2038 2039 2040 2041 2042 2043 2044 2045 2046 2047 2048 2049 2050 2051 2052 2053 2054 2055 2056 2057 2058 2059 2060 2061 2062 2063 2064 2065 2066 2067 2068 2069 2070 2071 2072 2073 2074 2075 2076 2077 2078 2079 2080 2081 2082 2083 2084 2085 2086 2087 2088 2089 2090 2091 2092 2093 2094 2095 2096 2097 2098 2099 2100 2101 2102 2103 2104 2105 2106 2107 2108 2109 2110 2111 2112 2113 2114 2115 2116 2117 2118 2119 2120 2121 2122 2123 2124 2125 2126 2127 2128 2129 2130 2131 2132 2133 2134 2135 2136 2137 2138 2139 2140 2141 2142 2143 2144 2145 2146 2147 2148 2149 2150 2151 2152 2153 2154 2155 2156 2157 2158 2159 2160 2161 2162 2163 2164 2165 2166 2167 2168 2169 2170 2171 2172 2173 2174 2175 2176 2177 2178 2179 2180 2181 2182 2183 2184 2185 2186 2187 2188 2189 2190 2191 2192 2193 2194 2195 2196 2197 2198 2199 2200 2201 2202 2203 2204 2205 2206 2207 2208 2209 2210 2211 2212 2213 2214 2215 2216 2217 2218 2219 2220 2221 2222 2223 2224 2225 2226 2227 2228 2229 2230 2231 2232 2233 2234 2235 2236 2237 2238 2239 2240 2241 2242 2243 2244 2245 2246 2247 2248 2249 2250 2251 2252 2253 2254 2255 2256 2257 2258 2259 2260 2261 2262 2263 2264 2265 2266 2267 2268 2269 2270 2271 2272 2273 2274 2275 2276 2277 2278 2279 2280 2281 2282 2283 2284 2285 2286 2287 2288 2289 2290 2291 2292 2293 2294 2295 2296 2297 2298 2299 2300 2301 2302 2303 2304 2305 2306 2307 2308 2309 2310 2311 2312 2313 2314 2315 2316 2317 2318 2319 2320 2321 2322 2323 2324 2325 2326 2327 2328 2329 2330 2331 2332 2333 2334 2335 2336 2337 2338 2339 2340 2341 2342 2343 2344 2345 2346 2347 2348 2349

$$[1.7 \times 10^{-10} \text{ mol/L}] \times 0.001 \text{ L} = 1.7 \times 10^{-13} \text{ mol} = 0.3 \text{ ng}$$

27.

There are no more.

[illegible]

104-105-106-107-108-109-110-111-112-113-114-115-116-117-118-119-120-121-122-123-124-125-126-127-128-129-130-131-132-133-134-135-136-137-138-139-140-141-142-143-144-145-146-147-148-149-150-151-152-153-154-155-156-157-158-159-160-161-162-163-164-165-166-167-168-169-170-171-172-173-174-175-176-177-178-179-180-181-182-183-184-185-186-187-188-189-190-191-192-193-194-195-196-197-198-199-200-201-202-203-204-205-206-207-208-209-210-211-212-213-214-215-216-217-218-219-220-221-222-223-224-225-226-227-228-229-230-231-232-233-234-235-236-237-238-239-240-241-242-243-244-245-246-247-248-249-250-251-252-253-254-255-256-257-258-259-260-261-262-263-264-265-266-267-268-269-270-271-272-273-274-275-276-277-278-279-280-281-282-283-284-285-286-287-288-289-290-291-292-293-294-295-296-297-298-299-300-301-302-303-304-305-306-307-308-309-310-311-312-313-314-315-316-317-318-319-320-321-322-323-324-325-326-327-328-329-330-331-332-333-334-335-336-337-338-339-340-341-342-343-344-345-346-347-348-349-350-351-352-353-354-355-356-357-358-359-360-361-362-363-364-365-366-367-368-369-370-371-372-373-374-375-376-377-378-379-380-381-382-383-384-385-386-387-388-389-390-391-392-393-394-395-396-397-398-399-400-401-402-403-404-405-406-407-408-409-410-411-412-413-414-415-416-417-418-419-420-421-422-423-424-425-426-427-428-429-430-431-432-433-434-435-436-437-438-439-440-441-442-443-444-445-446-447-448-449-450-451-452-453-454-455-456-457-458-459-460-461-462-463-464-465-466-467-468-469-470-471-472-473-474-475-476-477-478-479-480-481-482-483-484-485-486-487-488-489-490-491-492-493-494-495-496-497-498-499-500-501-502-503-504-505-506-507-508-509-510-511-512-513-514-515-516-517-518-519-520-521-522-523-524-525-526-527-528-529-530-531-532-533-534-535-536-537-538-539-540-541-542-543-544-545-546-547-548-549-550-551-552-553-554-555-556-557-558-559-560-561-562-563-564-565-566-567-568-569-570-571-572-573-574-575-576-577-578-579-580-581-582-583-584-585-586-587-588-589-590-591-592-593-594-595-596-597-598-599-600-601-602-603-604-605-606-607-608-609-610-611-612-613-614-615-616-617-618-619-620-621-622-623-624-625-626-627-628-629-630-631-632-633-634-635-636-637-638-639-640-641-642-643-644-645-646-647-648-649-650-651-652-653-654-655-656-657-658-659-660-661-662-663-664-665-666-667-668-669-670-671-672-673-674-675-676-677-678-679-680-681-682-683-684-685-686-687-688-689-690-691-692-693-694-695-696-697-698-699-700-701-702-703-704-705-706-707-708-709-710-711-712-713-714-715-716-717-718-719-720-721-722-723-724-725-726-727-728-729-730-731-732-733-734-735-736-737-738-739-740-741-742-743-744-745-746-747-748-749-750-751-752-753-754-755-756-757-758-759-760-761-762-763-764-765-766-767-768-769-770-771-772-773-774-775-776-777-778-779-780-781-782-783-784-785-786-787-788-789-790-791-792-793-794-795-796-797-798-799-800-801-802-803-804-805-806-807-808-809-810-811-812-813-814-815-816-817-818-819-820-821-822-823-824-825-826-827-828-829-830-831-832-833-834-835-836-837-838-839-840-841-842-843-844-845-846-847-848-849-850-851-852-853-854-855-856-857-858-859-860-861-862-863-864-865-866-867-868-869-870-871-872-873-874-875-876-877-878-879-880-881-882-883-884-885-886-887-888-889-890-891-892-893-894-895-896-897-898-899-900-901-902-903-904-905-906-907-908-909-910-911-912-913-914-915-916-917-918-919-920-921-922-923-924-925-926-927-928-929-930-931-932-933-934-935-936-937-938-939-940-941-942-943-944-945-946-947-948-949-950-951-952-953-954-955-956-957-958-959-960-961-962-963-964-965-966-967-968-969-970-971-972-973-974-975-976-977-978-979-980-981-982-983-984-985-986-987-988-989-990-991-992-993-994-995-996-997-998-999-1000-1001-1002-1003-1004-1005-1006-1007-1008-1009-1010-1011-1012-1013-1014-1015-1016-1017-1018-1019-1020-1021-1022-1023-1024-1025-1026-1027-1028-1029-1030-1031-1032-1033-1034-1035-1036-1037-1038-1039-1040-1041-1042-1043-1044-1045-1046-1047-1048-1049-1050-1051-1052-1053-1054-1055-1056-1057-1058-1059-1060-1061-1062-1063-1064-1065-1066-1067-1068-1069-1070-1071-1072-1073-1074-1075-1076-1077-1078-1079-1080-1081-1082-1083-1084-1085-1086-1087-1088-1089-1090-1091-1092-1093-1094-1095-1096-1097-1098-1099-1100-1101-

[illegible][illegible]

• • •

1. The first part of the paper is devoted to a review of the literature on the topic of the role of the state in the development of the economy. It is found that the state has played a significant role in the development of the economy in many countries, particularly in the case of developing countries. The state has been able to mobilize resources, provide infrastructure, and create a favorable environment for investment and growth.

1. The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \int_0^x \frac{1}{1+t^2} dt.$$

It is known that

$$f(x) = \arctan x.$$

The second part of the paper is devoted to the study of the properties of the function $F(x)$ defined by the equation

$$F(x) = \int_0^x \frac{1}{1+t^2} dt.$$

$$F(x) = \arctan x.$$

$$F(x) = \arctan x.$$

It is known that

$$F(x) = \arctan x.$$

It is known that

The third part of the paper is devoted to the study of the properties of the function $G(x)$ defined by the equation

$$G(x) = \int_0^x \frac{1}{1+t^2} dt.$$

$$G(x) = \arctan x.$$

$$G(x) = \arctan x.$$

$$G(x) = \arctan x.$$

The fourth part of the paper is devoted to the study of the properties of the function $H(x)$ defined by the equation

$$H(x) = \int_0^x \frac{1}{1+t^2} dt.$$

As another example, consider a function whose domain is $\{k: k \geq 4\}$, say the function a such that, for each $k \geq 4$, $a_k = k^2 + 1$. Let's find a Σ -expression for the sum of the first five values of a .

$$\begin{aligned}
 a_4 &= 4^2 + 1 = b_1 = (1+3)^2 + 1 \\
 a_5 &= 5^2 + 1 = b_2 = (2+3)^2 + 1 \\
 a_6 &= 6^2 + 1 = b_3 = (3+3)^2 + 1 \\
 a_7 &= 7^2 + 1 = b_4 = (4+3)^2 + 1 \\
 a_8 &= 8^2 + 1 = b_5 = (5+3)^2 + 1
 \end{aligned}$$

$$\sum_{p=1}^5 [(p+3)^2 + 1]$$

And, now, let's find a Σ -expression for the sum of three consecutive values of a starting with the tenth.

$$\begin{aligned}
 a_{13} &= 13^2 + 1 = c_1 = (1+12)^2 + 1 \\
 a_{14} &= 14^2 + 1 = c_2 = (2+12)^2 + 1 \\
 a_{15} &= 15^2 + 1 = c_3 = (3+12)^2 + 1
 \end{aligned}$$

$$\sum_{p=1}^3 [(p+12)^2 + 1]$$

*

In the equations implicit in lines 1-8 on page 8-36 students may see a short cut illustrated:

$$\sum_{k=0}^4 (2k+3) = \sum_{p=0+1}^{4+1} [2(\underbrace{p-1}_{\text{cloud}}) + 3] \qquad \sum_{p=3}^7 (2p+3) = \sum_{p=3-2}^{7-2} [2(\underbrace{p+2}_{\text{cloud}}) + 3]$$

[They will probably discover this short cut while doing some of the exercises of Part A on page 8-37.] The short cut is justified, later, by Theorem 137 on page 8-44.

It is a well known fact that the human mind is capable of receiving and storing information in a very efficient manner. This is the reason why we are able to remember things that we have learned or experienced. The human mind is a very powerful tool, and it is the only one that we have. It is the only one that can think, feel, and create. It is the only one that can learn from its mistakes and grow from its experiences. It is the only one that can love and be loved. It is the only one that can hope and dream. It is the only one that can make a difference in the world.

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Answers for Exercises 3 and 4 of Part C are on TC[8-34].

*

It may be helpful to give additional examples to illustrate the second-to-last sentence on page 8-35. Consider a function whose domain is $\{k: k \geq -5\}$, say the function a such that, for each $k \geq -5$, $a_k = 3k + 17$. Suppose that we wish to write a Σ -expression for the sum of the first seven values of a . Since we know how to write such expressions for sums of consecutive values, starting with the first, of functions whose domain is I^+ , what we need to do is to find a function b whose consecutive values are precisely those of a .

$$\begin{array}{rclcl}
 a_{-5} & = & 3 \cdot -5 + 17 & = & b_1 = 3(1-6) + 17 \\
 a_{-4} & = & 3 \cdot -4 + 17 & = & b_2 = 3(2-6) + 17 \\
 a_{-3} & = & 3 \cdot -3 + 17 & = & b_3 = 3(3-6) + 17 \\
 a_{-2} & = & 3 \cdot -2 + 17 & = & b_4 = 3(4-6) + 17 \\
 a_{-1} & = & 3 \cdot -1 + 17 & = & b_5 = 3(5-6) + 17 \\
 a_0 & = & 3 \cdot 0 + 17 & = & b_6 = 3(6-6) + 17 \\
 a_1 & = & 3 \cdot 1 + 17 & = & b_7 = 3(7-6) + 17 \\
 \hline
 & & & & \sum_{p=1}^7 [3(p-6) + 17]
 \end{array}$$

If we wish to write a Σ -expression for the sum of three consecutive values of the function a , beginning with the twentieth, we need to find a function c whose domain is I^+ and such that

$$\begin{array}{rclcl}
 a_{14} & = & 3 \cdot 14 + 17 & = & c_1 = 3(1+13) + 17 \\
 a_{15} & = & 3 \cdot 15 + 17 & = & c_2 = 3(2+13) + 17 \\
 a_{16} & = & 3 \cdot 16 + 17 & = & c_3 = 3(3+13) + 17 \\
 \hline
 & & & & \sum_{p=1}^3 [3(p+13) + 17]
 \end{array}$$

The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \frac{1}{2} (f(x-1) + f(x+1)) + \frac{1}{2} (f(x-2) + f(x+2)) + \dots$$

It is shown that the function $f(x)$ is periodic with period 2.

It is also shown that the function $f(x)$ is continuous and that it satisfies the functional equation

$$f(x) = \frac{1}{2} (f(x-1) + f(x+1)) + \frac{1}{2} (f(x-2) + f(x+2)) + \dots$$

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Answers for Part A.

1. $(3 \cdot 0 - 4) + (3 \cdot 1 - 4) + (3 \cdot 2 - 4) + (3 \cdot 3 - 4) + (3 \cdot 4 - 4) + (3 \cdot 5 - 4)$
 2. $(3 \cdot -2 + 2) + (3 \cdot -1 + 2) + (3 \cdot 0 + 2) + (3 \cdot 1 + 2) + (3 \cdot 2 + 2) + (3 \cdot 3 + 2)$
 3. $(3 \cdot 3 - 13) + (3 \cdot 4 - 13) + (3 \cdot 5 - 13) + (3 \cdot 6 - 13) + (3 \cdot 7 - 13) + (3 \cdot 8 - 13)$
- [Note that the Σ -expressions in Exercises 1, 2, and 3 are equivalent. This is a foreshadowing of Part D, and of Theorem 137 which appears on page 8-44.]
4. $(2^2 - 1) + (3^2 - 1) + (4^2 - 1) + (5^2 - 1) + (6^2 - 1)$
 5. $(31 - 2 \cdot 15) + (31 - 2 \cdot 16) + (31 - 2 \cdot 17)$
 6. $1 + 1 + 1 + 1 + 1 + 1 + 1$ [This is preparation for Part B.]
 7. $(10 - 10) + (11 - 10)$
 8. $(10 - 10)$
 9. 0

*

Answers for Part B.

- | | | | | |
|-------|------|------|----------------|------|
| 1. 4 | 2. 4 | 3. 4 | 4. 4 | 5. 9 |
| 6. 10 | 7. 9 | 8. 0 | 9. $k - j + 1$ | |

*

Answers for Part C.

1. $\sum_{p=1}^q 6p = 6 \sum_{p=1}^q p = 6 \frac{q(q+1)}{2} = 3q(q+1)$
2. $\sum_{p=1}^n 5p = 5 \sum_{p=1}^n p = 5 \frac{n(n+1)}{2}$

Answers for Part D.

$$1. \sum_{p=2}^6 (3 - 4p); \quad \sum_{p=1}^5 [3 - 4(p + 1)], \quad \left[\text{or: } \sum_{p=1}^5 (-1 - 4p) \right];$$

$$\sum_{p=5}^9 [3 - 4(p - 3)], \quad \left[\text{or: } \sum_{p=5}^9 (15 - 4p) \right];$$

$$\sum_{i=-2}^2 [3 - 4(i + 4)], \quad \left[\text{or: } \sum_{i=-2}^2 (-13 - 4i) \right]; \text{ etc.}$$

$$2. \sum_{p=1}^6 \frac{1}{2p}; \quad \sum_{p=9}^{14} \frac{1}{2(p - 8)}, \quad \left[\text{or: } \sum_{p=9}^{14} \frac{1}{2p - 16} \right];$$

$$\sum_{i=0}^5 \frac{1}{2(i + 1)}, \quad \left[\text{or: } \sum_{i=0}^5 \frac{1}{2i + 2} \right]; \quad \sum_{i=-3}^2 \frac{1}{2(i + 4)}, \quad \left[\text{or: } \sum_{i=-3}^2 \frac{1}{2i + 8} \right]; \text{ etc.}$$

$$3. \sum_{p=3}^6 \frac{p - 2}{p(p + 1)}; \quad \sum_{p=4}^7 \frac{p - 3}{(p - 1)p}; \quad \sum_{i=0}^3 \frac{i + 1}{(i + 3)(i + 4)};$$

$$\sum_{i=-5}^{-2} \frac{i + 6}{(i + 8)(i + 9)}; \text{ etc.}$$

*

Let $f(x) = x^2 + 2x + 1$. Find $f'(x)$.

$$f(x) = x^2 + 2x + 1$$

$$f'(x) = 2x + 2$$

$$\frac{d}{dx} \left(\frac{1}{x^2} \right) = -\frac{2}{x^3}$$

$$\frac{d}{dx} x^{-2} = -2x^{-3}$$

$$\frac{d}{dx} \left(\frac{1}{x^2} \right) = -\frac{2}{x^3}$$

$$\frac{d}{dx} \left(\frac{1}{x^2} \right) = -\frac{2}{x^3}$$

$$\frac{d}{dx} \left(\frac{1}{x^2} \right) = -\frac{2}{x^3}$$

Answers for Part E.

[There are many correct answers for each exercise.] [Recall, in

Exercise 1, that ' $\sum_{p=3}^5 2p + 12$ ' is an abbreviation for ' $\left(\sum_{p=3}^5 2p\right) + 12$ '.]

$$1. \sum_{p=3}^5 2p + 2 \cdot 6 = \sum_{p=3}^6 2p$$

$$2. (3 \cdot 7 + 1) + \sum_{p=8}^{11} (3p + 1) = \sum_{p=7}^{11} (3p + 1)$$

$$3. \sum_{p=5}^{13} (3p - 11)$$

$$4. \sum_{p=4}^{10} p^2$$

$$5. \sum_{k=-3}^9 (2k + 5)$$

$$6. \sum_{p=1}^{n+1} 2p(p + 2)$$

$$7. \sum_{p=1}^{12} (2p^2 - 1)$$

$$8. \sum_{p=1}^{12} (p + 2)$$

$$9. \sum_{k=-5}^{14} (5k + 31)$$

$$10. \sum_{p=5}^{n+2} p$$

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Proof of Theorem 133:

(i) By the recursive definition on page 8-36 and the pm0,

$$\sum_{i=j}^{j-1} ca_i = 0 = c \cdot 0 = c \sum_{i=j}^{j-1} a_i.$$

(ii) Suppose that, for some $k \geq j - 1$, $\sum_{i=j}^k ca_i = c \sum_{i=j}^k a_i$. Then, by the recursive definition [since $k \geq j - 1$],

$$\begin{aligned} \sum_{i=j}^{k+1} ca_i &= \sum_{i=j}^k ca_i + ca_{k+1} = c \sum_{i=j}^k a_i + ca_{k+1} \\ &= c \left[\sum_{i=j}^k a_i + a_{k+1} \right] = c \sum_{i=j}^{k+1} a_i, \end{aligned}$$

by the ldpma and the recursive definition. Hence,

$$\forall_{k \geq j-1} \left[\sum_{i=j}^k ca_i = c \sum_{i=j}^k a_i \Rightarrow \sum_{i=j}^{k+1} ca_i = c \sum_{i=j}^{k+1} a_i \right].$$

(iii) It follows from (i) and (ii), by Theorem 114, that

$$\forall_{k \geq j-1} \sum_{i=j}^k ca_i = c \sum_{i=j}^k a_i. \text{ Consequently, } \forall_x \forall_j \dots$$

The answer for Exercise 1 of Part A is in the COMMENTARY for page 8-40.

Answers for Part A.

[Exercise 1 is on page 8-39.]

$$1. \sum_{p=1}^q 2p = 2 \sum_{p=1}^q p = 2 \frac{q(q+1)}{2} = q(q+1).$$

$$\text{Consequently, } \forall_n \sum_{p=1}^n 2p = n(n+1).$$

*

The theorems of Exercises 1 and 2 could be proved as well by using the theorem in the first of the Exploration Exercises on page 8-31, rather than Theorem 133. Make sure that students realize that the earlier theorem is a consequence of Theorem 133 and is, in fact, the consequence one needs in proving the theorems of Exercises 1 and 2.

$$(1) \quad [\text{Theorem 133}] \quad [\text{theorem}]$$

$$(2) \quad \forall_{k \geq 1} \sum_{i=1}^k ca_i = c \sum_{i=1}^k a_i \quad [(1)]$$

$$(3) \quad q \geq 0 \quad [\text{theorem}]$$

$$(4) \quad \sum_{i=1}^q ca_i = c \sum_{i=1}^q a_i \quad [(2), (3)]$$

$$(5) \quad \forall_x \forall_n \sum_{i=1}^n xa_i = x \sum_{i=1}^n a_i \quad [(1) - (4)]$$

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$$2. (a) \sum_{p=1}^q \frac{p(p+1)}{2} = \frac{1}{2} \sum_{p=1}^q p(p+1) = \frac{1}{2} \cdot \frac{q(q+1)(q+2)}{3} = \frac{q(q+1)(q+2)}{6}.$$

Consequently,

$$(b) \sum_{p=1}^q (p - \frac{1}{2}) = \sum_{p=1}^q \frac{2p-1}{2} = \frac{1}{2} \sum_{p=1}^q (2p-1) = \frac{1}{2} \cdot q^2 = \frac{q^2}{2}.$$

Consequently,

$$3. \sum_{i=j}^k x = \sum_{i=j}^k x \cdot 1 = x \sum_{i=j}^k 1 = x(k-j+1)$$

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Answers for Part B.

$$1. (a) \sum_{p=1}^{10} \frac{1}{10} \cdot \frac{p}{10} = \frac{1}{10^2} \sum_{p=1}^{10} p = \frac{10 \cdot 11}{200} = \frac{11}{20}$$

$$(b) \forall_n \sum_{p=1}^n \frac{p}{n^2} = \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2n} \quad \left[= \frac{1}{2} + \frac{1}{2n} \right]$$

The remaining answers for Part B are in the COMMENTARY for page 8-41.

The following table shows the results of the experiments conducted on the 10th of May 1904. The first column gives the number of the experiment, the second column the number of the subject, the third column the number of the trial, the fourth column the number of the error, the fifth column the number of the correct answer, and the sixth column the number of the total number of trials.

| Experiment | Subject | Trial | Error | Correct | Total |
|------------|---------|-------|-------|---------|-------|
| 1 | 1 | 1 | 1 | 1 | 2 |
| 2 | 2 | 2 | 2 | 2 | 4 |
| 3 | 3 | 3 | 3 | 3 | 6 |
| 4 | 4 | 4 | 4 | 4 | 8 |
| 5 | 5 | 5 | 5 | 5 | 10 |
| 6 | 6 | 6 | 6 | 6 | 12 |
| 7 | 7 | 7 | 7 | 7 | 14 |
| 8 | 8 | 8 | 8 | 8 | 16 |
| 9 | 9 | 9 | 9 | 9 | 18 |
| 10 | 10 | 10 | 10 | 10 | 20 |

The results of the experiments show that the number of errors decreases as the number of trials increases. This is evident from the fact that the number of errors is always less than the number of correct answers. The number of errors is also always less than the number of trials. This is evident from the fact that the number of errors is always less than the number of correct answers.

The results of the experiments also show that the number of correct answers increases as the number of trials increases. This is evident from the fact that the number of correct answers is always greater than the number of errors. The number of correct answers is also always greater than the number of trials. This is evident from the fact that the number of correct answers is always greater than the number of errors.

2. We assume from the figure that $a < 0 < b$. So, from Exercise 1, the area-measure of the region below the parabola is

$$\frac{b^3}{3} + \frac{(-a)^3}{3}.$$

The total shaded region is bounded by a trapezoid. Hence, its area-measure is

$$\frac{1}{2}(b - a)(b^2 + a^2).$$

So, the ratio of the area-measure of the upper shaded region to that of the lower shaded region is

$$\begin{aligned} \frac{\frac{1}{2}(b - a)(b^2 + a^2) - \left(\frac{b^3}{3} - \frac{a^3}{3}\right)}{\frac{b^3}{3} - \frac{a^3}{3}} &= \frac{3(b - a)(b^2 + a^2) - 2(b^3 - a^3)}{2(b^3 - a^3)} \\ &= \frac{(b - a)[3(b^2 + a^2) - 2(b^2 + ba + a^2)]}{2(b^3 - a^3)} = \frac{(b - a)^3}{2(b^3 - a^3)}. \end{aligned}$$

*

Students should see that, for $j = 1$ and $k = 2$, Theorem 134 on page 8-42 implies:

$$(a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1 + b_2)$$

which is an instance of the sum rearrangement theorem.

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bottom of page 8-42: In practice, one would probably skip most steps and write:

$$\sum_{p=1}^{12} (3 - 4p) = 3 \cdot 12 - 4 \cdot \frac{12 \cdot 13}{2} = \dots$$

The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function, and its value is determined by the initial condition $f(0) = 1$.

In the second part, we consider the problem of finding the maximum value of the function $f(x)$ on the interval $[0, 1]$. It is shown that the maximum value is attained at $x = 0$ and is equal to 1. This result is obtained by using the fact that $f(x)$ is a constant function.

The third part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function, and its value is determined by the initial condition $f(0) = 1$.

In the fourth part, we consider the problem of finding the maximum value of the function $f(x)$ on the interval $[0, 1]$. It is shown that the maximum value is attained at $x = 0$ and is equal to 1. This result is obtained by using the fact that $f(x)$ is a constant function.

The fifth part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function, and its value is determined by the initial condition $f(0) = 1$.

In the sixth part, we consider the problem of finding the maximum value of the function $f(x)$ on the interval $[0, 1]$. It is shown that the maximum value is attained at $x = 0$ and is equal to 1. This result is obtained by using the fact that $f(x)$ is a constant function.

The seventh part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function, and its value is determined by the initial condition $f(0) = 1$.

6. $1/(m + 1)$ [To check this guess students would need at least partial knowledge of a summation theorem for the sequence of m th powers. The needed knowledge is contained in (*) on page 8-204.]

*

There is much that can be said in connection with Part B. For one thing, although students have, in Unit 6, adopted a definition of area-measure for polygonal regions according to which the area-measure of the triangular region of Exercises 1, 2, and 3 is $1/2$, they are probably unacquainted with any definition of area-measure which applies to the regions discussed in Exercises 4, 5, and 6. As indicated in the COMMENTARY for page 6-349 of Unit 6, even in the case of circular regions such an extension of the definition of area-measure requires the introduction of some limit concept. In this connection, you can tell students that the process of approximations on which they based their guesses in Exercises 4, 5, and 6 is actually used, in calculus, as the basis of a definition of area-measure. So, their guesses turn out to be correct by definition!

*

Answers for Part ☆C.

$$1. \frac{a}{10} \left(\frac{a}{10} \right)^2 + \frac{a}{10} \left(\frac{2a}{10} \right)^2 + \frac{a}{10} \left(\frac{3a}{10} \right)^2 + \dots + \frac{a}{10} \left(\frac{10a}{10} \right)^2 = \left(\frac{a}{10} \right)^3 \sum_{p=1}^{10} p^2 = 0.385a^3;$$

$$\forall_n \sum_{p=1}^n \frac{a^3 p^2}{n^3} = \frac{a^3 (n+1)(2n+1)}{6n^2} \left[= a^3 \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \right) \right]; \text{ etc.}$$

A good guess for the area-measure of the region bounded by the graphs of ' $y = x^2$ ', ' $y = 0$ ', and ' $x = a$ ' is $a^3/3$.

The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \sum_{n=0}^{\infty} \frac{f_n(x)}{n!}$, where $f_n(x)$ is a function of the n -th order. It is shown that the function $f(x)$ is analytic in the region $|x| < 1$ and that it satisfies the differential equation $f'(x) = f(x)$. The second part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \sum_{n=0}^{\infty} \frac{f_n(x)}{n!}$, where $f_n(x)$ is a function of the n -th order. It is shown that the function $f(x)$ is analytic in the region $|x| < 1$ and that it satisfies the differential equation $f'(x) = f(x)$.

1. Introduction

The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \sum_{n=0}^{\infty} \frac{f_n(x)}{n!}$, where $f_n(x)$ is a function of the n -th order. It is shown that the function $f(x)$ is analytic in the region $|x| < 1$ and that it satisfies the differential equation $f'(x) = f(x)$.

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2. Conclusion

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| | | | | | |
|--------|-------------|-------|---------|-----------|-------------|
| 3. (a) | n | 10 | 100 | 1000 | 10000 |
| | U_n | 0.385 | 0.33835 | 0.3338335 | 0.333383335 |
| | L_n | 0.285 | 0.32835 | 0.3328335 | 0.333283335 |
| | $U_n - L_n$ | 0.1 | 0.01 | 0.001 | 0.0001 |

3. (b) $1/3$

$$5. \quad 1. \quad (a) \quad \sum_{p=1}^{10} \frac{p^3}{10^4} = \frac{1}{10^4} \left(\frac{10 \cdot 11}{2} \right)^2 = 0.3025$$

$$1. \quad (b) \quad \forall_n \sum_{p=1}^n \frac{p^3}{n^4} = \frac{1}{n^4} \left(\frac{n(n+1)}{2} \right)^2 = \frac{(n+1)^2}{4n^2} \quad \left[= \frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2} \right]$$

$$2. \quad (a) \quad \sum_{p=1}^9 \frac{p^3}{10^4} = \frac{1}{10^4} \left(\frac{9 \cdot 10}{2} \right)^2 = 0.2025$$

$$2. \quad (b) \quad \forall_n \sum_{p=1}^{n-1} \frac{p^3}{n^4} = \frac{1}{n^4} \left(\frac{(n-1)n}{2} \right)^2 = \frac{(n-1)^2}{4n^2} \quad \left[= \frac{1}{4} - \frac{1}{2n} + \frac{1}{4n^2} \right]$$

| | | | | | |
|--------|-------------|--------|----------|------------|--------------|
| 3. (a) | n | 10 | 100 | 1000 | 10000 |
| | U_n | 0.3025 | 0.255025 | 0.25050025 | 0.2500500025 |
| | L_n | 0.2025 | 0.245025 | 0.24950025 | 0.2499500025 |
| | $U_n - L_n$ | 0.1 | 0.01 | 0.001 | 0.0001 |

(b) $1/4$

$$2. (a) \sum_{p=1}^9 \frac{p}{10^2} = \frac{9 \cdot 10}{200} = \frac{9}{20}$$

$$(b) \forall_n \sum_{p=1}^{n-1} \frac{p}{n^2} = \frac{n-1}{2n} \left[= \frac{1}{2} - \frac{1}{2n} \right]$$

3. (a)

| n | 10 | 100 | 1000 | 10000 |
|-------------|-------|---------|-----------|-------------|
| U_n | 11/20 | 101/200 | 1001/2000 | 10001/20000 |
| L_n | 9/20 | 99/200 | 999/2000 | 9999/20000 |
| $U_n - L_n$ | 1/10 | 1/100 | 1/1000 | 1/10000 |

(b) The sequences of overestimates and underestimates "approach" $1/2$ as the term number increases. A good guess for the area-measure of the triangular region is $1/2$.

$$4. 1. (a) \frac{1}{10} \left(\frac{1}{10} \right)^2 + \frac{1}{10} \left(\frac{2}{10} \right)^2 + \frac{1}{10} \left(\frac{3}{10} \right)^2 + \dots + \frac{1}{10} \left(\frac{10}{10} \right)^2$$

$$= \sum_{p=1}^{10} \frac{p^2}{10^3} = \frac{1}{10^3} \sum_{p=1}^{10} p^2 = \frac{1}{10^3} \cdot \frac{10 \cdot 11 \cdot 21}{6} = 0.385$$

$$1. (b) \forall_n \sum_{p=1}^n \frac{p^2}{n^3} = \frac{(n+1)(2n+1)}{6n^2} \left[= \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \right]$$

$$2. (a) \sum_{p=1}^9 \frac{p^2}{10^3} = \frac{9 \cdot 10 \cdot 19}{10^3 \cdot 6} = 0.285$$

$$2. (b) \forall_n \sum_{p=1}^{n-1} \frac{p^2}{n^3} = \frac{1}{n^3} \cdot \frac{(n-1)n(2n-1)}{6} = \frac{(n-1)(2n-1)}{6n^2} \left[= \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} \right]$$

(9) $\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2}{dt^2} \right) = \frac{1}{2} \frac{d^3}{dt^3}$

(10) $\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2}{dt^2} \right) = \frac{1}{2} \frac{d^3}{dt^3}$

(11) $\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2}{dt^2} \right) = \frac{1}{2} \frac{d^3}{dt^3}$

(12) $\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2}{dt^2} \right) = \frac{1}{2} \frac{d^3}{dt^3}$

(13) $\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2}{dt^2} \right) = \frac{1}{2} \frac{d^3}{dt^3}$

(14) $\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2}{dt^2} \right) = \frac{1}{2} \frac{d^3}{dt^3}$

(15) $\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2}{dt^2} \right) = \frac{1}{2} \frac{d^3}{dt^3}$

*

Answers for Part E.

$$1. \sum_{p=1}^9 (1 + 2p) = \sum_{p=1}^9 1 + 2 \sum_{p=1}^9 p = 9 + 2 \cdot \frac{9 \cdot 10}{2} = 99$$

$$\begin{aligned} 2. \sum_{p=1}^{100} (p^2 + p) &= \sum_{p=1}^{100} p^2 + \sum_{p=1}^{100} p = \frac{100 \cdot 101 \cdot 201}{6} + \frac{100 \cdot 101}{2} \\ &= \frac{100 \cdot 101}{2} \left(\frac{201}{3} + 1 \right) = 343400 \end{aligned}$$

$$\begin{aligned} 3. \sum_{p=1}^{50} (p^3 - p) &= \sum_{p=1}^{50} p^3 - \sum_{p=1}^{50} p = \left(\frac{50 \cdot 51}{2} \right)^2 - \frac{50 \cdot 51}{2} = \frac{50 \cdot 51}{2} \left(\frac{50 \cdot 51}{2} - 1 \right) \\ &= 1624350 \end{aligned}$$

$$\begin{aligned} 4. \sum_{p=1}^n p(p + k) &= \sum_{p=1}^n p^2 + k \sum_{p=1}^n p = \frac{n(n+1)(2n+1)}{6} + \frac{kn(n+1)}{2} \\ &= \frac{n(n+1)(2n+1+3k)}{6} \end{aligned}$$

5. and 6. [See lower part of page 8-43.]

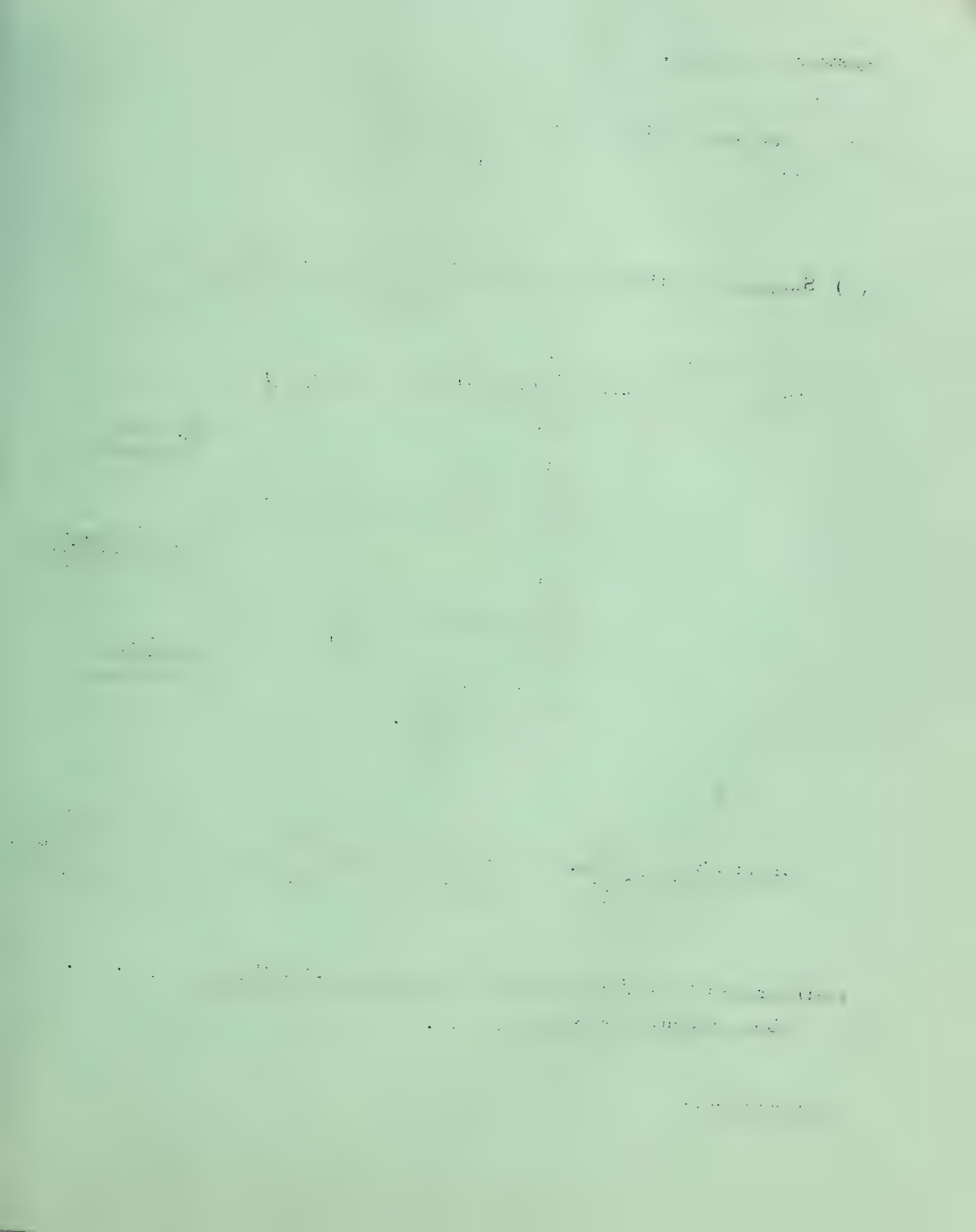
*

Students should see, on page 8-44, that for $j = 1$, $j_1 = 1$, and $k = 3$, Theorem 135 implies:

$$a_1 + a_2 + a_3 = a_1 + (a_2 + a_3)$$

which is an instance of the associative principle for addition.

For $j_1 = j$, Theorem 135 implies Theorem 136.



Answer for Part D.

$$(i) \quad \sum_{i=j}^{j-1} (a_i + b_i) = 0 = 0 + 0 = \sum_{i=j}^{j-1} a_i + \sum_{i=j}^{j-1} b_i$$

$$(ii) \quad \text{Suppose that [for some } k \geq j-1] \quad \sum_{i=j}^k (a_i + b_i) = \sum_{i=j}^k a_i + \sum_{i=j}^k b_i.$$

$$\begin{aligned} \text{Then} \quad \sum_{i=j}^{k+1} (a_i + b_i) &= \sum_{i=j}^k (a_i + b_i) + (a_{k+1} + b_{k+1}) \\ &= \sum_{i=j}^k a_i + \sum_{i=j}^k b_i + (a_{k+1} + b_{k+1}) \\ &= \left(\sum_{i=j}^k a_i + a_{k+1} \right) + \left(\sum_{i=j}^k b_i + b_{k+1} \right) \\ &= \sum_{i=j}^{k+1} a_i + \sum_{i=j}^{k+1} b_i. \end{aligned} \quad \left. \begin{array}{l} \text{inductive} \\ \text{hypothesis} \\ \text{sum rearrange-} \\ \text{ment theorem} \\ \text{recursive} \\ \text{definition} \end{array} \right\}$$

$$\text{Hence, } \forall_{k \geq j-1} \left(\sum_{i=j}^k (a_i + b_i) = \sum_{i=j}^k a_i + \sum_{i=j}^k b_i \Rightarrow \sum_{i=j}^{k+1} (a_i + b_i) = \sum_{i=j}^{k+1} a_i + \sum_{i=j}^{k+1} b_i \right).$$

(iii) From (i) and (ii) it follows, by Theorem 114, that $\forall_{k \geq j-1} \dots$

Consequently, $\forall_j \forall_{k \geq j-1} \dots$

$$f(x) = \frac{1}{2} \left(\frac{1}{x} + \frac{1}{x+1} \right) = \frac{1}{2} \left(\frac{x+1}{x(x+1)} + \frac{x}{x(x+1)} \right) = \frac{1}{2} \left(\frac{x+1+x}{x(x+1)} \right) = \frac{1}{2} \left(\frac{2x+1}{x(x+1)} \right) = \frac{2x+1}{2x(x+1)}$$

$$f(x) = \frac{1}{2} \left(\frac{1}{x} + \frac{1}{x+1} \right) = \frac{1}{2} \left(\frac{x+1}{x(x+1)} + \frac{x}{x(x+1)} \right) = \frac{1}{2} \left(\frac{x+1+x}{x(x+1)} \right) = \frac{1}{2} \left(\frac{2x+1}{x(x+1)} \right) = \frac{2x+1}{2x(x+1)}$$

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$$5. \sum_{i=-3}^{18} (i + 5) = \sum_{i=1}^{22} [(i - 4) + 5] = \sum_{i=1}^{22} (i + 1) = 275$$

$$6. \sum_{j=-5}^{12} (11 - 2j) = \sum_{j=1}^{18} [11 - 2(j - 6)] = \sum_{j=1}^{18} (23 - 2j) = 72$$

$$7. \sum_{p=4}^{15} \left(\frac{1}{2}p - 5\right) + \sum_{p=16}^{20} \left(\frac{1}{2}p - 5\right) = \sum_{p=4}^{20} \left(\frac{1}{2}p - 5\right) = \sum_{p=1}^{17} \left[\frac{1}{2}(p + 3) - 5\right]$$

$$= \frac{1}{2} \sum_{p=1}^{17} (p - 7) = 17$$

$$8. \sum_{p=1}^{50} (4 - 3p) + \sum_{q=1}^{50} (3q - 4) = \sum_{p=1}^{50} (4 - 3p) + - \sum_{q=1}^{50} (4 - 3q) = 0$$

where $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$. The function $f(x)$ is defined as follows: $f(x) = \frac{1}{2}x^2 + \frac{1}{2}x$. The function $f(x)$ is defined as follows: $f(x) = \frac{1}{2}x^2 + \frac{1}{2}x$.

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Answers for Part A.

$$1. \sum_{p=1}^6 [(p+4) - 1]$$

$$2. \sum_{p=7}^{12} [(p-2) - 1]$$

$$3. \sum_{i=0}^5 [(i+5) - 1]$$

$$4. \sum_{i=-2}^3 [(i+7) - 1]$$

$$5. \sum_{p=1}^{11} [3 - 2(p-5)]$$

$$6. \sum_{p=2}^{n+1} \frac{1}{p}$$

*

Answers for Part B.

$$1. \sum_{p=1}^{14} (4p+3) = 4 \sum_{p=1}^{14} p + \sum_{p=1}^{14} 3 = 4 \frac{14 \cdot 15}{2} + 3 \cdot 14 = 462$$

$$2. \sum_{p=9}^{20} p = \sum_{p=1}^{12} (p+8) = \sum_{p=1}^{12} p + \sum_{p=1}^{12} 8 = 174$$

$$\text{or: } \sum_{p=9}^{20} p = \sum_{p=1}^{20} p - \sum_{p=1}^8 p = \frac{20 \cdot 21}{2} - \frac{8 \cdot 9}{2} = 174$$

$$3. -255$$

$$4. \sum_{p=8}^{19} (2+4p) = \sum_{p=1}^{12} [2+4(p+7)] = \sum_{p=1}^{12} (30+4p) = 2 \sum_{p=1}^{12} (15+2p) = 672$$

Answers for Part C.

1. Since $\sum_{p=1}^n (2p + 1) = 2 \cdot \frac{n(n+1)}{2} + n = n(n+2)$, the given equation is equivalent to:

$$n(n+2) = 15$$

This equation has the root 3.

$$\begin{aligned} 2. \quad \sum_{p=1}^q (2p - 1)^2 &= \sum_{p=1}^q (4p^2 - 4p + 1) = 4 \cdot \frac{q(q+1)(2q+1)}{6} - 4 \cdot \frac{q(q+1)}{2} + q \\ &= \frac{q[2(q+1)(2q+1) - 6(q+1) + 3]}{3} \\ &= \frac{q(4q^2 + 6q + 2 - 6q - 6 + 3)}{3} \\ &= \frac{q(4q^2 - 1)}{3} \end{aligned}$$

$$\begin{aligned} 3. \quad \sum_{i=-3}^j 2i &= \sum_{i=1}^{j+4} 2(i-4) = 2 \sum_{i=1}^{j+4} (i-4) = 2 \left[\frac{(j+4)(j+5)}{2} - 4(j+4) \right] \\ &= (j+4)(j-3) \end{aligned}$$

[The use made of Theorem 137 requires the restriction ' $j \geq -4$ '. The use of Theorem 131 requires the stronger restriction ' $j+4 \geq 1$ '. So, the preceding algebra justifies the theorem of Exercise 3. Actually, one could prove the theorem with the restricted quantifier ' $\forall_{k \geq -4}$ '.]

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$$P = \frac{(1 + \rho) \dots (1 + \rho) (1 + \rho)}{1 + \rho} \dots$$

$$\frac{1}{1 + \rho} \dots \frac{1}{1 + \rho} \dots \frac{1}{1 + \rho} \dots$$

$$\frac{1}{1 + \rho} \dots \frac{1}{1 + \rho} \dots \frac{1}{1 + \rho} \dots$$

$$\frac{1}{1 + \rho} \dots \frac{1}{1 + \rho} \dots \frac{1}{1 + \rho} \dots$$

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$$\frac{1}{1 + \rho} \dots \frac{1}{1 + \rho} \dots \frac{1}{1 + \rho} \dots$$

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4. The area-measure of the exposed surface of the p th cube [$1 \leq p \leq n$], counting down from the top, is

$$\underbrace{4p^2}_{\substack{\uparrow \\ \text{sides}}} + \underbrace{[p^2 - (p-1)^2]}_{\substack{\uparrow \\ \text{top}}}, \quad \text{or} \quad 4p^2 + 2p - 1.$$

$$\begin{aligned} \sum_{p=1}^n (4p^2 + 2p - 1) &= 4 \cdot \frac{n(n+1)(2n+1)}{6} + 2 \cdot \frac{n(n+1)}{2} - n \\ &= \frac{n[2(n+1)(2n+1) + 3(n+1) - 3]}{3} \\ &= \frac{n[4n^2 + 6n + 2 + 3n + 3 - 3]}{3} \\ &= \frac{n(n+2)(4n+1)}{3} \end{aligned}$$

[A quick-answer question to follow this:

If the weight of the topmost cube is 1 ounce,
what is the weight of the step-pyramid?

Ans: $\left(\frac{n(n+1)}{2}\right)^2$ ounces [Theorem 131d.]]

An interesting variation is to make the step-pyramid out of n rectangular blocks of dimensions 1 by 2 by 3, 2 by 3 by 4, ..., n by $n+1$ by $n+2$, with each block resting on one of its largest faces. In this case, the area-measure of the exposed surface of the topmost block is

$$2[1 \cdot 2 + 1 \cdot 3] + 2 \cdot 3$$

and, for $1 < p \leq n$, the area-measure of the exposed surface of the p th block is

$$\underbrace{2[p(p+1) + p(p+2)]}_{\text{sides}} + \underbrace{[(p+1)(p+2) - p(p+1)]}_{\text{top}}.$$

So [rewriting the first case as ' $2[1 \cdot 2 + 1 \cdot 3] + [2 \cdot 3 - 1 \cdot 2] + 1 \cdot 2$ '], the area-measure of the total exposed surface is

$$\sum_{p=1}^n \left[2[p(p+1) + p(p+2)] + [(p+1)(p+2) - p(p+1)] \right] + 1 \cdot 2$$

--that is, $\frac{2(n+1)(2n+1)(n+3)}{3}$.

[Again, the volume of the step-pyramid is easily found, this time by using Theorem 132d. Ask for its weight if the topmost block weighs 6 ounces.]

In finding the area-measure of the total exposed surface of either of the step-pyramids discussed above, students may discover that the area-measure of the exposed horizontal surface is the same as the area-measure of the top surface of the bottom block. [Take a birds-eye view!] So, for the first pyramid, the area measure of the total exposed surface is

$$\sum_{p=1}^n 4p^2 + n^2,$$

and that for the second is

$$\sum_{p=1}^n 2[p(p+1) + p(p+2)] + (n+1)(n+2).$$

Students who notice this are well on their way to discovering Theorem 138 on page 8-53. In the case of the first pyramid this theorem [with $a_p = (p-1)^2$] tells us that

$$\sum_{p=1}^n [p^2 - (p-1)^2] = n^2 - (1-1)^2 = n^2.$$

In the case of the second pyramid the theorem [with $a_p = p(p+1)$] tells us that

$$\sum_{p=1}^n [(p+1)(p+2) - p(p+1)] = (n+1)(n+2) - 1 \cdot 2;$$

so, the area-measure of the exposed horizontal surface is $(n+1)(n+2) - 1 \cdot 2 + 1 \cdot 2$.

★5. (a) Suppose that the integers are $p, p+1, \dots, p+(q-1)$, so that

$$\begin{aligned} 100 &= \sum_{i=p}^{p+q-1} i = \frac{(p+q-1)(p+q)}{2} - \frac{(p-1)p}{2} \quad [\text{Theorem 135}] \\ &= \frac{q(2p+q-1)}{2}. \end{aligned}$$

So, find positive integers p and q [with $q \geq 2$] such that

$$(1) \quad q(2p+q-1) = 200.$$

Evidently q and $2p+q-1$ are factors of 200 and, since their product is 200 and $q < 2p+q-1$, $q < \sqrt{200} \doteq 14$. Since $q \geq 2$ and $200 = 2^3 \cdot 5^2$, the only possibilities for q are 2, 4, 5, 8, and 10. Noting that, by (1), $(200/q) - q = 2p - 1$ [an odd positive integer], we solve the problem as follows:

| q | $200/q$ | $2p-1$ | p |
|-----|---------|--------|-----|
| 2 | 100 | 98 | — |
| 4 | 50 | 46 | — |
| 5 | 40 | 35 | 18 |
| 8 | 25 | 17 | 9 |
| 10 | 20 | 10 | — |

So, the only sets of two or more consecutive positive integers with sum 100 are $\{18, 19, 20, 21, 22\}$ and $\{9, 10, 11, 12, 13, 14, 15, 16\}$.

(b) A similar analysis, with (1) replaced by:

$$(2) \quad q(2p+q-1) = 2 \cdot 16$$

shows that there are no sets of two or more consecutive positive integers whose sum is 16.



Exercise ☆5 can be supplemented in several ways. One way is to remove the restriction that the sought-for integer be positive. This amounts to removing the restriction that $q < 2p + q - 1$. So, the additional solutions for part (a) can be obtained by extending the table to test the remaining positive integral factors [other than 1]--20, 25, 40, 50, 100, and 200--of 200. Doing so, one obtains three additional solutions for part (a), $\{-8, -7, \dots, 16\}$, $\{-17, -16, \dots, 22\}$ and $\{-99, -98, \dots, 100\}$. Each of the first two are obtained by tacking on a 0-sum set of integers to one of the solutions of the original problem. The third is obtained in a similar way by starting with the set $\{100\}$. Evidently, this generalization of the given problem is rather trivial--once one has found solutions in positive integers one can obtain the solutions without the restrictions in a trivial manner. For example, for part (b), removing the restriction results in gaining a single solution $\{-15, -14, \dots, 16\}$.

A more interesting investigation is into the difference between 100 and 16 which leads to part (a) having solutions while part (b) does not. Given a positive integer n , how can one tell whether n is the sum of two or more consecutive positive integers? More explicitly, how many sets of consecutive positive integers have the sum n ?

To answer the second question [and, with it, the first] we consider the equation:

$$(3) \qquad q(2p + q - 1) = 2n$$

As in part (a), positive integral solutions p and q of the equation correspond with sets $\{p, p+1, \dots, p+(q-1)\}$ of q consecutive integers whose sum is n . Note, first, that if q is an even number then, since $2p - 1$ is odd, $2p + q - 1$ is odd, and that if q is odd then $2p + q - 1$ is even. So, of the two positive integers q and $2p + q - 1$ whose product is $2n$, one is an odd factor of $2n$ and, so, is an odd factor of n . [That, for each n , an odd factor of $2n$ is a factor of n , is a consequence of Theorem 128.] Suppose, now, that m is any odd factor of n . Solving equation (3) for 'p':

$$p = \frac{n}{q} - \frac{q-1}{2}$$

we see that (3) will have a solution (p, q) for which $q = m$ if and only if $\frac{n}{m} - \frac{m-1}{2}$ is a positive integer. Now, since m is a factor of n , $\frac{n}{m}$ is an integer; and, since m is odd, $(m-1)/2$ is an integer. Hence,

$\frac{n}{m} - \frac{m-1}{2}$ is, in any case, an integer. It is a positive integer if and only if $(m-1)/2 < n/m$. So, if m is a "small" odd factor of n then n is the sum of m consecutive positive integers. [We must discard the smallest odd factor, 1, of n , if we wish to restrict our search to sets of two or more consecutive positive integers whose sum is n .]

There remains the possibility that, not q , but $2p + q - 1$ is m . In this case, for a solution (p, q) of (3), we must have $q = 2n/m$ and [substituting in (3) and solving for 'p'] $p = \frac{m+1}{2} - \frac{n}{m}$. Since m is an odd factor of n , $\frac{m+1}{2}$ is a positive integer and, as in the previous case, $\frac{m+1}{2} - \frac{n}{m}$ is an integer, and is positive if and only if $\frac{m+1}{2} > \frac{n}{m}$. The latter is the case if and only if $\frac{m+1}{2} - 1 \geq \frac{n}{m}$ -- that is, if and only if $\frac{m-1}{2} \geq \frac{n}{m}$. So, if m is a "large" odd factor of n then n is the sum of $2n/m$ consecutive positive integers.

Summarizing, we see that the number of sets of two or more consecutive positive integers whose sum is n is precisely the number of odd factors of n other than 1.

Applying the results obtained above to the solution of Exercise ☆5 we see that since the odd factors of 100 [other than 1] are 5 and 25, there are just two sets of two or more positive integers whose sum is 100. Since $(5-1)/2 < 100/5$, one of these sets has 5 members, starting with 18. Since $(25+1)/2 \geq 100/25$, we see that the other has $(2 \cdot 100)/25$ members, starting with 9. On the other hand, since 16 has no odd factor other than 1, it follows that 16 is not the sum of two or more consecutive positive integers.

A similar problem is that of determining how many sets of consecutive odd integers have a given sum n . Using a similar analysis, based on the equation:

$$n = \sum_{i=p}^{p+q-1} (2i-1) = (p+q-1)^2 - (p-1)^2 = q(2p+q-2),$$

one sees that the number of such sets is the number of factors m of n such that $m^2 \leq n$ and either m and n/m are both even or m and n/m are both odd. [Incidentally, this is the same problem as that of determining the number of ways in which a given positive integer n is a difference of squares of positive integers.]

Answers for Part ☆D are in the COMMENTARY for page 8-47.

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- (ii) Suppose that $\forall_{m \leq p} F_m = af_{m-2} + bf_{m-1}$. Since, for $p > 1$, $F_{p+1} = F_{p-1} + F_p$, it follows [since $p-1 \in I^+$ and $p-1 \leq p$, and $p \leq p$] that $F_{p+1} = [af_{p-3} + bf_{p-2}] + [af_{p-2} + bf_{p-1}]$
 $= a[f_{p-3} + f_{p-2}] + b[f_{p-2} + f_{p-1}]$
 $= af_{p-1} + bf_p$, by the r.d. (**) for f .

On the other hand, $F_{1+1} = b = a0 + b1 = af_0 + bf_1$.

Hence,

$$\forall_n [\forall_{m \leq n} F_m = af_{m-2} + bf_{m-1} \Rightarrow F_{n+1} = af_{n-1} + bf_n].$$

- (iii) From (i) and (ii) it follows, by the SPMI, that

$$\forall_n F_n = af_{n-2} + bf_{n-1}.$$

The preceding SPMI-proof for the theorem of Exercise 5 is not much simpler than the proof given earlier in the COMMENTARY. It is somewhat more "natural"--less tricky. The true strength of the SPMI appears when one has to go back more than two steps to find material to justify the conclusion ' $n+1 \in S$ '.

As pointed out above, it is sufficient for our purpose to show merely that the SPMI is a theorem. The easiest way to do this is via the least number theorem [Theorem 108].

Suppose that [for a given set S of real numbers] $1 \in S$ and that, for each n , if each $m \leq n$ belongs to S then $n + 1 \in S$. And, suppose that $p \notin S$. Since $1 \in S$ and $p \notin S$, it follows that $p \neq 1$. Hence, by Theorem 104, $p > 1$ and, by (I^+_1) and Theorem 105, $p - 1 \in I^+$. So, by hypothesis, if each $m \leq p - 1$ belongs to S then $p - 1 + 1 \in S$. But, since $p - 1 + 1 = p$ and $p \notin S$, this is not the case. Hence, there is a positive integer--say, q --such that $q \leq p - 1$ and $q \notin S$. Since, by Theorems 90 and 92, $q < p$, it follows that if $p \notin S$ then there is a $q < p$ such that $q \notin S$. Hence, if the set of those positive integers which do not belong to S is nonempty then it is a nonempty set of positive integers which has no least member. Since, by Theorem 108, there is no such set, it follows that the set of positive integers which do not belong to S is empty--that is, that each positive integer belongs to S .

[For theoretical purposes it is often desirable to restate the SPMI as:

$$\forall_S [\forall_n [\forall_{m < n} m \in S \Rightarrow n \in S] \Rightarrow \forall_n n \in S]$$

That this is possible is due to the fact that the instance for $n = 1$ of the antecedent is equivalent to ' $1 \in S$ ' and that ' $\forall_{n > 1} [\forall_{m < n} m \in S \Rightarrow n \in S]$ ' is equivalent to ' $\forall_n [\forall_{m < n+1} m \in S \Rightarrow n + 1 \in S]$ '. However, this apparently simpler statement is seldom of advantage in giving proofs. In order to establish its antecedent one generally has to separate proofs of its two "parts", and this amounts to using our form of the SPMI.]

We shall now illustrate the use of the SPMI by proving:

$$\forall_n F_n = af_{n-2} + bf_{n-1} \quad [\text{Exercise 5 of Part } \star D]$$

$$(i) \quad F_1 = a = a1 + b0 = af_{-1} + bf_0$$

However, the strength of the SPMI consists, essentially, in simplifying proofs rather than, as one might at first think, in making it possible to prove theorems which cannot be proved by using the PMI. In fact, any theorem which can be proved by using the SPMI can also be proved by using the FMI and some previously proved theorems. One way to show that this is the case is to show how to transform an SPMI-proof into a PMI-proof. This is somewhat more complicated than the reverse process which has been described above, and we shall discuss it only briefly. But, it is very easy to justify the use of the SPMI by showing that it is a theorem. This we shall do by deriving it from previously proved theorems.

To transform an SPMI-proof into a PMI-proof one proceeds as follows. Given S , let $S^* = \{p: \forall_{m \leq p} m \in S\}$. Now, $1 \in S^*$ if and only if $\forall_{m \leq 1} 1 \in S$. But, since $\forall_{m \leq 1} m = 1$ [Theorems 104 and 93], the latter is the case if $1 \in S$. Since part (i) of the given SPMI-proof shows that $1 \in S$, this part of the proof can be transformed into a proof of ' $1 \in S^*$ '.

Part (ii) of the given SPMI-proof amounts to deriving ' $p + 1 \in S$ ' from the inductive hypothesis ' $p \in S^*$ '. So, from this inductive hypothesis one can derive ' $p \in S^*$ and $p + 1 \in S$ '--that is, ' $\forall_{m \leq p} m \in S$ and $p + 1 \in S$ '. But, since $\forall_{m \leq p+1} (m \leq p \text{ or } m = p + 1)$ [by the only-if-part of Theorem 107b], it follows from the latter that $\forall_{m \leq p+1} m \in S$ --that is, that $p + 1 \in S^*$. So, part (ii) of the SPMI-proof can be transformed into a proof of ' $\forall_n [n \in S^* \Rightarrow n + 1 \in S^*]$ '.

Now, from the conclusions of the transformed parts (i) and (ii) it follows, by the PMI, that $\forall_n n \in S^*$ --that is, that $\forall_n \forall_{m \leq n} m \in S$. But, since $\forall_n n \leq n$, it follows from this that $\forall_n n \in S$.

So, by considerably complicating the given SPMI-proof, one can obtain a PMI-proof of the same conclusion.

the first of these is the fact that the
the second is the fact that the
the third is the fact that the

the fourth is the fact that the
the fifth is the fact that the
the sixth is the fact that the

the seventh is the fact that the
the eighth is the fact that the
the ninth is the fact that the

the tenth is the fact that the
the eleventh is the fact that the
the twelfth is the fact that the

the thirteenth is the fact that the
the fourteenth is the fact that the
the fifteenth is the fact that the

★In the COMMENTARY for Exercise 4 of Part ★D it was remarked that it might be possible to establish the theorem of that exercise by using a more powerful kind of inductive proof. This is, indeed, the case and we shall now discuss the strong principle of mathematical induction [SPMI]:

$$\forall_S [(1 \in S \text{ and } \forall_n [\forall_{m \leq n} m \in S \Rightarrow n + 1 \in S]) \Rightarrow \forall_n n \in S]$$

which is useful in proving this and other theorems.

The strength of the SPMI resides in the fact that while its conditional has the same conclusion, ' $\forall_n n \in S$ ', as does the PMI [that is, as does (I_3^+)], its antecedent contains the component:

$$(a) \quad \forall_n [\forall_{m \leq n} m \in S \Rightarrow n + 1 \in S]$$

instead of:

$$(b) \quad \forall_n [n \in S \Rightarrow n + 1 \in S]$$

A proof using the SPMI is just like a proof using the PMI except that in part (ii) one establishes (a) [for the given set S], rather than (b). In either case, part (ii) consists mainly of deriving the same conclusion, ' $p + 1 \in S$ ', from an inductive hypothesis. In the case of the SPMI the inductive hypothesis is ' $\forall_{m \leq p} m \in S$ ', while in the case of the PMI the inductive hypothesis is ' $p \in S$ '. Since [because $\forall_n n \leq n$] the second of these hypotheses is an immediate consequence of the first, it follows that whenever one can use the PMI, he can just as easily use the SPMI. [Take part (ii) of a PMI-proof beginning with the sentence 'Suppose that $p \in S$.' Replace this sentence by:

Suppose that $\forall_{m \leq p} m \in S$. Then, since $p \leq p$, $p \in S$.

The result is part (ii) of an SPMI-proof of the same theorem.]

So, whenever one can prove (b), it is just as easy to prove (a). But, sometimes it is easier to prove (a) than to prove (b). This is, of course, because the hypothesis ' $\forall_{m \leq p} m \in S$ ' gives one more to work with than does ' $p \in S$ '.

•

(ii) Suppose [for some $k \geq j - 1$] that $\sum_{i=j}^k a_i = \sum_{i=j+j_1}^{k+j_1} a_{i-j_1}$. By the

r. d. for Σ -notation, it follows that $\sum_{i=j}^{k+1} a_i = \sum_{i=j+j_1}^{k+j_1} a_{i-j_1} + a_{k+1}$

$$= \sum_{i=j+j_1}^{k+j_1} a_{i-j_1} + a_{k+j_1+1-j_1} = \sum_{i=j+j_1}^{k+j_1+1} a_{i-j_1} = \sum_{i=j+j_1}^{k+1+j_1} a_{i-j_1}.$$

Etc.

*

4. The theorem can be proved in several ways. One is by a method which might have been discovered in solving Exercise 4 of Part C on page 8-46 [see COMMENTARY] and which is discussed on page 8-52. The proof may be motivated as follows: First, either by experiences like that furnished by the first problem on page 8-1, or by one's familiarity with elementary algebra, he is led to think of the fact that, for each p , $(p+1)^2 = p^2 + (2p+1)$ and, so, $2p+1 = (p+1)^2 - p^2$. This suggests adding up differences.

$$\begin{array}{rcl} \sum_{p=1}^n (2p+1) & = & \sum_{p=1}^n [(p+1)^2 - p^2] = \begin{array}{r} \cancel{2^2} - 1^2 \\ + \cancel{3^2} - \cancel{2^2} \\ + \cancel{4^2} - \cancel{3^2} \\ \vdots \\ + \cancel{n^2} - \cancel{(n-1)^2} \\ + (n+1)^2 - \cancel{n^2} \\ \hline (n+1)^2 - 1^2 \end{array} \end{array}$$

This is the procedure which is formally justified by Th. 138 on 8-53.

*

$$2. (i) \text{ For } j_1 \geq j - 1, \quad \sum_{i=j}^{j_1} a_i = \sum_{i=j}^{j_1} a_i + 0 = \sum_{i=j}^{j_1} a_i + \sum_{i=j_1+1}^{j_1} a_i.$$

(ii) Suppose [for some $j_1 \geq j - 1$ and some $k \geq j_1$] that

$$\sum_{i=j}^k a_i = \sum_{i=j}^{j_1} a_i + \sum_{i=j_1+1}^k a_i. \text{ Since } k \geq j - 1, \text{ it follows, by the r.d.}$$

$$\text{for } \Sigma\text{-notation, that } \sum_{i=j}^{k+1} a_i = \sum_{i=j}^k a_i + a_{k+1}. \text{ Hence, } \sum_{i=j}^{k+1} a_i$$

$$= \sum_{i=j}^{j_1} a_i + \sum_{i=j_1+1}^k a_i + a_{k+1} = \sum_{i=j}^{j_1} a_i + \sum_{i=j_1+1}^{k+1} a_i, \text{ again, by the r.d.}$$

for Σ -notation, since $k \geq j_1$. Consequently, for $j_1 \geq j - 1$,

$$\forall_{k \geq j_1} \left[\sum_{i=j}^k a_i = \sum_{i=j}^{j_1} a_i + \sum_{i=j_1+1}^k a_i \Rightarrow \sum_{i=j}^{k+1} a_i = \sum_{i=j}^{j_1} a_i + \sum_{i=j_1+1}^{k+1} a_i \right].$$

(iii) From (i) and (ii) it follows, by Theorem 114, that, for $j_1 \geq j - 1$,

$$\forall_{k \geq j_1} \sum_{i=j}^k a_i = \sum_{i=j}^{j_1} a_i + \sum_{i=j_1+1}^k a_i.$$

Consequently, the theorem follows [by two applications of the test-pattern principle].

$$3. (i) \quad \sum_{i=j}^{j-1} a_i = 0 = \sum_{i=j+j_1}^{j-1+j_1} a_{i-j_1} \quad [\text{by the r.d. for } \Sigma\text{-notation, since } j-1+j_1 = j+j_1-1]$$

(ii) Suppose that $\forall_m [(m|f_p \text{ and } m|f_{p+1}) \Rightarrow m = 1]$, and suppose that $q|f_{p+1}$ and $q|f_{p+2}$. Since $f_{p+2} = f_p + f_{p+1}$, and since $q|f_{p+1}$ and $q|f_{p+2}$, it follows, by Theorem 126e, that $q|f_p$. Since $q|f_p$ and $q|f_{p+1}$ it follows by the inductive hypothesis that $q = 1$.

Hence, if $q|f_{p+1}$ and $q|f_{p+2}$ then $q = 1$ --

$$\forall_m [(m|f_{p+1} \text{ and } m|f_{p+2}) \Rightarrow m = 1].$$

Consequently, $\forall_n [\forall_m [(m|f_n \text{ and } m|f_{n+1}) \Rightarrow m = 1] \Rightarrow \forall_m [(m|f_{n+1} \text{ and } m|f_{n+2}) \Rightarrow m = 1]]$.

(iii) From (i) and (ii) the theorem follows by the PMI.

2. (i) Since $F_1 = a$ and $F_2 = b$, it follows that $\text{HCF}(F_1, F_2) = \text{HCF}(a, b)$.

(ii) Since $F_{p+2} = F_p + F_{p+1}$, it follows that F_{p+1} and F_{p+2} have the same common factors as do F_p and F_{p+1} . So, F_{p+1} and F_{p+2} have the same highest common factor as do F_p and F_{p+1} . Hence, if $\text{HCF}(F_p, F_{p+1}) = \text{HCF}(a, b)$ then $\text{HCF}(F_{p+1}, F_{p+2}) = \text{HCF}(a, b)$. Consequently, $\forall_n [\text{HCF}(F_n, F_{n+1}) = \text{HCF}(a, b) \Rightarrow \text{HCF}(F_{n+1}, F_{n+2}) = \text{HCF}(a, b)]$.

(iii) From (i) and (ii) the theorem follows by the PMI.

*

Answers for Part ★F.

1. By Theorem 135, for $k \geq j$, $\sum_{i=j}^k a_i = \sum_{i=j}^j a_i + \sum_{i=j+1}^k a_i$. But, by the

recursive definition for Σ -notation, $\sum_{i=j}^j a_i = \sum_{i=j}^{j-1} a_i + a_j = 0 + a_j = a_j$.

$$\sum_{p=1}^q F_p = \sum_{p=1}^q [af_{p-2} + bf_{p-1}] = a \sum_{p=1}^q f_{p-2} + b \sum_{p=1}^q f_{p-1}$$

$$= a \sum_{p=-1}^{q-2} f_p + b \sum_{p=0}^{q-1} f_p = a[1 + 0 + \sum_{p=1}^{q-2} f_p] + b[0 + \sum_{p=1}^{q-1} f_p]$$

$$= a[1 + 0 + f_{q-1}] + b[0 + f_{q-1}] = af_q + bf_{q+1} - b = F_{q+2} - b$$

[In the sequence of steps above we use, first, the theorem of Exercise 5, then Theorems 133 and 134, next Theorem 137, then Theorem 136, next the theorem of Exercise 2 on page 8-25, then algebra, and, finally, the theorem of Exercise 5. Note that the use of Theorem 136 requires the restriction ' $q \geq 2$ ' and that of the theorem of Exercise 2 requires the restriction ' $q \geq 3$ '. So, to complete the proof

$$\text{one must add: } \sum_{p=1}^1 F_p = F_1 = a = a + b - b = F_3 - b; \quad \sum_{p=1}^2 F_p = F_1 + F_2$$

$= F_3 = a + b = a + 2b - b = F_4 - b]$ The preceding proof is interesting in that it makes use of several important theorems. However, the theorem in question can be established more easily, just as that of Exercise 2 on page 8-25, by using Theorem 130.

*

Answers for Part ★E.

1. The proof that ' $\forall_n \text{HCF}(f_n, f_{n+1}) = 1$ ' amounts to proving the theorem of Exercise 2 with ' F ' replaced by ' f ' and ' a ' and ' b ' by ' 1 ', and then using the theorem ' $\text{HCF}(1, 1) = 1$ '. So, for this it is sufficient to recast the answer given below for Exercise 2. Alternatively, one could prove ' $\forall_n \forall_m [(m|f_n \text{ and } m|f_{n+1}) \Rightarrow m = 1]$ '. Here is a proof of this theorem:

(i) if $q|f_1$ and $q|f_2$ then, since $f_1 = 1$, $q = 1$.

To make the instance for $n = 1$ meaningful, both ' f_{-1} ' and ' f_0 ' must be defined; the instance for $n = 2$ requires that ' f_0 ' be defined. Exercise 2 has shown that if we accept the definitions ' $f_{-1} = 1$ ' and ' $f_0 = 0$ ' then the usual Fibonacci recursion equation will yield the terms of the Fibonacci sequence f . This suggests giving a new meaning to ' f ' by replacing the recursive definition (*) on page 8-46 of the text by:

$$(**) \quad \begin{cases} f_{-1} = 1 \\ f_0 = 0 \\ \forall_n f_n = f_{n-2} + f_{n-1} \end{cases}$$

Using this one can, as in the answer for Exercise 4, prove the generalization:

$$(\dagger\dagger\dagger\dagger) \quad \forall_n (F_n = af_{n-2} + bf_{n-1} \text{ and } F_{n+1} = af_{n-1} + bf_n)$$

from which the desired theorem follows at once. Here is the proof [compare it with that of ($\dagger\dagger$), above]:

$$(i) \quad F_1 = a = a1 + b0 = af_{-1} + bf_0, \text{ and } F_2 = b = a0 + b1 = af_0 + bf_1.$$

$$(ii) \quad \text{Suppose that } F_p = af_{p-2} + bf_{p-1} \text{ and } F_{p+1} = af_{p-1} + bf_p.$$

$$\begin{aligned} \text{Then,} \quad F_{p+1} &= af_{p-1} + bf_p \text{ and } F_{p+2} = F_p + F_{p+1} \\ &= a[f_{p-2} + f_{p-1}] + b[f_{p-1} + f_p] = af_p + bf_{p+1}. \end{aligned}$$

$$\begin{aligned} \text{Hence,} \quad \forall_n [(F_n = af_{n-2} + bf_{n-1} \text{ and } F_{n+1} = af_{n-1} + bf_n) \\ \Rightarrow (F_{n+1} = af_{n-1} + bf_n \text{ and } F_{n+2} = af_n + bf_{n+1})]. \end{aligned}$$

(iii) From (i) and (ii) it follows, by the PMI, that ($\dagger\dagger\dagger\dagger$).

$$6. \quad \forall_n \sum_{p=1}^n F_p = F_{n+2} - b;$$

conclusion ' $F_{p+3} = af_{p+1} + bf_{p+2}$ '. Evidently, what is needed is some more powerful kind of inductive proof or some trick for using the usual kind of inductive proof in a more powerful way.

It is not difficult to discover how to adopt the second alternative. [The first will be discussed later in this COMMENTARY.] Instead of attempting to prove (†) directly, we prove the equivalent generalization:

$$(††) \quad \forall_n (F_{n+2} = af_n + bf_{n+1} \text{ and } F_{n+3} = af_{n+1} + bf_{n+2})$$

Here is the inductive proof:

$$(i) \quad \begin{aligned} F_3 &= F_1 + F_2 = a + b = a1 + b1 = af_1 + bf_2 \text{ and} \\ F_4 &= F_2 + F_3 = b + (a + b) = a1 + b2 = af_2 + bf_3 \end{aligned}$$

(ii) Suppose [inductive hypothesis] that

$$F_{p+2} = af_p + bf_{p+1} \text{ and } F_{p+3} = af_{p+1} + bf_{p+2}.$$

Then [by the inductive hypothesis, alone], $F_{p+3} = af_{p+1} + bf_{p+2}$ and [by the recursive definition for F , the inductive hypothesis, and the recursive definition for f] $F_{p+4} = F_{p+2} + F_{p+3} = [af_p + bf_{p+1}] + [af_{p+1} + bf_{p+2}] = a[f_p + f_{p+1}] + b[f_{p+1} + f_{p+2}] = af_{p+2} + bf_{p+3}$. Hence,

$$\forall_n [(F_{n+2} = af_n + bf_{n+1} \text{ and } F_{n+3} = af_{n+1} + bf_{n+2}) \Rightarrow (F_{n+3} = af_{n+1} + bf_{n+2} \text{ and } F_{n+4} = af_{n+2} + bf_{n+3})]$$

(iii) From (i) and (ii) it follows, by the PMI that (††).

5. The procedure illustrated in this exercise--that of extending a definition in order to simplify applications of it--is common in mathematics. [An earlier illustration is the extension of the definition of Σ -notation.] The theorem to be stated and proved is:

$$(†††) \quad \forall_n F_n = af_{n-2} + bf_{n-1}$$

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Answers for Part ★D [which begins on page 8-46].

1. 4, 6, 10, 16, 26, 42, 68, 110, 178, 288
2. 1, 0, 1, 1, 2, 3, 5, 8, 13, 21
3. The successive terms of f are those which one obtains by starting with the 3rd term of the sequence F of Exercise 2--that is, $\forall_n F_{n+2} = f_n$.
4. Comparing the terms of f and F ,

f : 1, 1, 2, 3, 5, 8, 13, ...

F : a , b , $a+b$, $a+b \cdot 2$, $a \cdot 2 + b \cdot 3$, $a \cdot 3 + b \cdot 5$, $a \cdot 5 + b \cdot 8$, ...

suggests the theorem:

$$(†) \quad \forall_n F_{n+2} = af_n + bf_{n+1}$$

[Notice that this is consistent with the answer for Exercise 3, and with that for Exercise 1. For example, $288 = 4 \cdot 21 + 6 \cdot 34$.] An inductive proof of the suggested theorem involves a new twist. Part (i) of such a proof will offer no trouble. However, consider part (ii). The inductive hypothesis will be ' $F_{p+2} = af_p + bf_{p+1}$ ' and, from this and the recursive definitions of F and f , one is to deduce ' $F_{p+3} = af_{p+1} + bf_{p+2}$ '. Starting out, one notes that, since, by the recursive definition of F , $F_{p+3} = F_{p+1} + F_{p+2}$, it follows from the inductive hypothesis that $F_{p+3} = F_{p+1} + [af_p + bf_{p+1}]$. Now, if one could replace ' F_{p+1} ' by ' $af_{p-1} + bf_p$ ', he would be able to conclude that $F_{p+3} = a[f_{p-1} + f_p] + b[f_p + f_{p+1}]$ and, using the recursive definition of f , that $F_{p+3} = a[f_{p-1} + f_p] + bf_{p+2}$. But, in the first place, ' $F_{p+1} = af_{p-1} + bf_p$ ' is not available to justify the required substitution, and, in fact, this equation leads [at present] to nonsense in the case $p = 1$ [but, see Exercise 5]. For this second reason [' f_0 ' not defined], one could not, even had he gotten so far, complete the derivation by replacing ' $f_{p-1} + f_p$ ' by ' f_{p+1} ' to obtain the desired

Answers for Miscellaneous Exercises.

$$\left[\begin{array}{l} \text{Easy:} \quad 1-6, 10-21, 23, 24, 26-29, 31, 32, 34, 36-41; \\ \text{Medium:} \quad 7-9, 22, 33, 35; \\ \text{Hard:} \quad 25, 30 \end{array} \right]$$

1. $P = 4\sqrt{K}$

2. (a) $\frac{7}{2x - y}$

(b) $\frac{5}{2a + 4b}$

3. $5\sqrt{3}$

4. $\frac{5}{3} \left[\frac{A}{B} = \frac{2}{3} \Rightarrow \frac{A}{B} + 1 = \frac{5}{3} \right]$

5. $C = \frac{AB + D}{B}$

6. $\frac{1}{2}$ and $\frac{1}{4}$

7. Suppose that s is the side-measure of the square and x is the measure of one of the legs of $\triangle PAQ$. Then,

$$K(\triangle PAQ) = \frac{x(s - x)}{2} = \frac{\frac{s^2}{4} - \left(\frac{s}{2} - x\right)^2}{2}$$

The maximum value of K is attained when $x = \frac{s}{2}$. In that case $K = \frac{s^2}{8}$.

8. 15 for 68 cents [Suppose that he had $2n$ apples. Then, he collected a total of $17(n/3) + 17(n/5)$ cents. So, he could have collected the same amount if he sold all at the rate of $2n/[17(n/3) + 17(n/5)]$ apples per cent--that is, at the rate of 15 apples for 68 cents. [Warn students in advance that the answer is not '4 for 17 cents'.]]

Students may like to figure out the least number of apples the merchant could have had. [Since n must be divisible by 3 and by 5, $2n$ is a multiple of 15.]

1. The first part of the paper is devoted to the study of the

properties of the function $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $x \in \mathbb{R}$.

2. The second part of the paper is devoted to the study of the

properties of the function $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $x \in \mathbb{C}$.

3. The third part of the paper is devoted to the study of the

properties of the function $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $x \in \mathbb{R}$.

4. The fourth part of the paper is devoted to the study of the

properties of the function $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $x \in \mathbb{C}$.

5. The fifth part of the paper is devoted to the study of the

properties of the function $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $x \in \mathbb{R}$.

6. The sixth part of the paper is devoted to the study of the

properties of the function $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $x \in \mathbb{C}$.

7. The seventh part of the paper is devoted to the study of the

properties of the function $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $x \in \mathbb{R}$.

9. 3 [Students will probably implicitly assume that the domain of 'x' is the set of integers. If they do, fine. For this case they know, from Unit 4, that $27^x = (3^3)^x = 3^{3x}$, and should see that $3^{2x+3} = 3^{3x}$ if $x = 3$. Later in this unit they will see how to prove that 3 is the only integral root of the given equation. In Unit 9 they will learn of nonintegral exponents and be able to show that 3 is the only real root of the equation.]
10. (D) [The absolute values of the differences are, in order, $1/6$, $1/48$, $1/33$, $1/66$, and $1/30$.]
11. 9

25. [All measures are treated as positive.] Since $x < a$, $y < b$, and $z < c$, $1/x > 1/a$, $1/y > 1/b$, and $1/z > 1/c$. So,

$$\begin{aligned}
 & \frac{1}{x} + \frac{1}{y} + \frac{1}{z} > \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \\
 \Leftrightarrow & \frac{xy + yz + zx}{xyz} > \frac{ab + bc + ca}{abc} \\
 \Leftrightarrow & \frac{xy + yz + zx}{xyz} > \frac{2(xy + yz + zx)}{abc} \quad \left. \vphantom{\frac{xy + yz + zx}{xyz}} \right\} \text{by hypothesis} \\
 \Leftrightarrow & \frac{1}{xyz} > \frac{2}{abc} \\
 \Leftrightarrow & xyz < \frac{1}{2} abc \\
 \Rightarrow & xyz \neq \frac{1}{2} abc
 \end{aligned}$$

Note that what has been shown is that "decreasing the dimensions of a rectangular solid increases the ratio of its area to its volume". This is why small people are likely to be more affected by changes in temperature than are large ones.

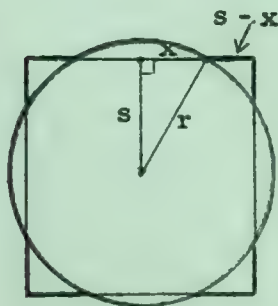
26. 2250

27. By the distance formula, the side-measures are $\sqrt{34}$, $\sqrt{34}$, and $\sqrt{68}$. So, by definition, the triangle is isosceles. [Since $\sqrt{34} + \sqrt{34} \neq \sqrt{68}$ and $\sqrt{34} + \sqrt{68} \neq \sqrt{34}$, the three points are noncollinear.] By the converse of the Pythagorean theorem, the triangle is a right triangle.

28. $36 \cos 63^\circ$ feet [$\doteq 16\frac{1}{3}$ feet]

29. gn/54 dollars

30.



Suppose that r is the radius of the circle and $2s$ is the side-measure of the square. Then, $100(s - x)/s$ is the required per cent. Now, $x^2 = r^2 - s^2$. Since, by hypothesis, $\pi r^2 = 4s^2$, $x^2 = (4 - \pi)s^2/\pi$. So, $x = s\sqrt{(4 - \pi)/\pi}$. Therefore, the required per cent is $100(1 - \sqrt{(4 - \pi)/\pi})$.

31. $(xy + uv)/(y + v)$ miles per hour.

12. 11

13. $K = \frac{1}{2}w(P - 2w)$

14. 15

15. 711 $[x + (x + 106) = 1528]$

16. (a) $\frac{5}{12}|x|$ (b) $\frac{97}{40}$ (c) $\frac{3p+4}{6p^2}$

17. (a) 2.5 (b) 10.1

18. $66\frac{2}{3}$ [A quick way to do this problem is to note that the altitude from A of $\triangle FAB$ is half as long as the altitude from C of $\triangle FCB$. So, the area-measure of $\triangle FCB$ is twice the area-measure of $\triangle FAB$ or $2/3$ that of $\triangle FABC$. A longer procedure involves simplifying the fraction:

$$\frac{\left(2 \cdot \frac{s}{2}\sqrt{3}\right)s}{\left(2 \cdot \frac{s}{2}\sqrt{3}\right)s + 2\left(\frac{1}{2} \cdot \frac{s}{2}\left[2 \cdot \frac{s}{2}\sqrt{3}\right]\right)}$$

Students should see both approaches.]

19. (a) $\frac{8a+6b}{3a}$

(b) $\frac{343t^3 - 112t}{49t^2 - 8}$

20. 4 [See discussion of Exercise 9 on TC[8-48]b.]

21. -2 $\left[\frac{1}{x} - \frac{1}{y} = \frac{y-x}{xy} = -\frac{x-y}{xy} = -\frac{24}{12}\right]$

22. $4/\sqrt{17}$ $[(x+y)^2 = (x^2 + y^2) + 2(xy) = 9 + 8 = 17. \text{ Since } x \in P \text{ and } y \in P, x+y \in P. \text{ So, } x+y = \sqrt{17}.]$

23. $(5x - y)/4$

24. $z = (x - y)/x$ $[z \neq 1, x \neq 0]$

$$36. -5 \text{ and } -2 \quad [(a+3)^2 + (a+3) - 2 = 0 \iff (a+3+2)(a+3-1) = 0]$$

$$37. 0, 6 \quad [(6+x)^2 - 2 = 6^2 + 2(6+x)]$$

$$38. 30$$

$$39. (a) 11x^3 + 3x^2 - 6x - 7$$

$$(b) 3(x+y)^3 - 2(x+y)^2 + 6(x+y) + 2$$

$$(c) 6x^3 - 44x^2 + 30x$$

$$40. (a) x^2 + 5x + 25$$

$$(b) x^4 + x^2 + 1$$

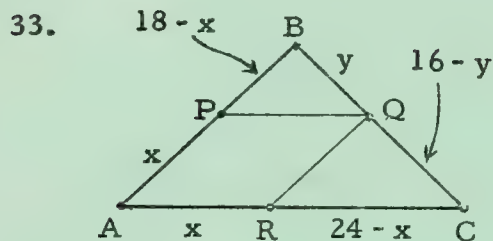
$$41. \text{ Since } P = 6s \text{ and } K = 2\left(\frac{1}{2} \cdot \frac{\sqrt{3}}{2} s(s+d)\right), \text{ and since } d = 2s, \text{ it follows}$$

$$\text{that } P = 3d \text{ and } K = \frac{3\sqrt{3}}{8} d^2.$$

$$32. \frac{2}{7}(x^2 - y^2)$$

$$(b) \frac{2}{3}(y^2 - y - 6)$$

$$(c) \frac{3}{8ab^4}$$



$$\text{Since } \overrightarrow{PQ} \parallel \overrightarrow{AC}, \frac{x}{18} = \frac{16 - y}{16}.$$

$$\text{Since } \overrightarrow{RQ} \parallel \overrightarrow{AB}, \frac{24}{x} = \frac{16}{y}.$$

$$\text{Hence, } \frac{x}{18} \cdot \frac{24}{x} = \frac{16 - y}{16} \cdot \frac{16}{y}$$

$$\text{Thus, } \frac{4}{3} = \frac{16 - y}{y}, \text{ and } y = \frac{48}{7}.$$

So, $AP = \frac{72}{7}$, $BQ = \frac{48}{7}$, and $CR = \frac{96}{7}$. [The proportion $\frac{24}{18} = \frac{16 - y}{y}$ follows immediately from the fact that \overrightarrow{AQ} bisects $\angle A$ and the theorem that an angle bisector of a triangle divides the opposite side into segments proportional to the including sides. See Exercise ☆29 on page 6-436 of Unit 6.]

$$34. \frac{100n}{n + m + p}$$

35. 1 [The problem is equivalent to finding the number of integers x such that $|x - 2| < x - 1$ and $|x - 3| < 2 - \frac{x}{2}$. Now, $|x - 2| < x - 1$ if and only if $-(x - 1) < x - 2 < x - 1$, and $|x - 3| < 2 - \frac{x}{2}$ if and only if $-(2 - \frac{x}{2}) < x - 3 < 2 - \frac{x}{2}$. So, our problem is to find the number of integers x such that $(x > 3/2 \text{ and } -2 < -1)$, and $(x/2 > 1 \text{ and } 3x/2 < 5)$ --that is, such that $x > 2$ and $x < \frac{10}{3}$. There is one such integer--3.]

[The exercise can also be solved by graphing. Doing so leads one to notice that any solutions (x, y) will be pairs of integers such that $x = y$, and $y > 1$ and $y < 4$. So, the only possibilities are $(2, 2)$ and $(3, 3)$. The first does not satisfy $|x - 3| < 2 - \frac{x}{2}$, but the second is a solution.]

In the early pages of this unit, students discovered theorems for continued sums by experimenting with special cases and looking for patterns [see page 8-1]. This is the manner in which Theorems 131 and 132 were discovered. With the help of Theorems 133-137, Theorems 131 and 132 were put to use in deriving other theorems for continued sums [see Part B on page 8-45] but this time without the experimenting-and-searching-for-patterns procedure. Although these latter theorems could have been discovered by experimenting, such a chore would have been difficult. [To see how difficult, try the step-pyramid problem (Exercise 4 on page 8-46) for, say, 100 cubes, on students who have not learned Theorems 131-137.] So, we now have some powerful machinery to use in solving rather difficult problems and in discovering theorems which are not at all obvious. The work on summation by differences which begins on page 8-52 will add another weapon to this problem-solving arsenal.

You may find it helpful to introduce this topic with a difficult-appearing problem and let students observe you at work solving it. Here is such a problem.

A step-pyramid is formed by stacking 100 thin rectangular plates. [If you and your class are acquainted with the Tower of Hanoi problem, use that as a setting.]

Each plate has a volume of 3 cubic inches. The plate at the very top has a base 4 inches by 5 inches, the next one is thinner and has a base 5 inches by 6 inches, the next is still thinner and has a base 6 inches by 7 inches, etc. How tall is the pyramid?

If you wish, you can let students struggle with this for a while. Then you solve it. We wish to compute

$$\frac{3}{4 \cdot 5} + \frac{3}{5 \cdot 6} + \frac{3}{6 \cdot 7} + \cdots + \frac{3}{(p+3)(p+4)} + \cdots + \frac{3}{103 \cdot 104},$$

that is,

$$\sum_{p=1}^{100} \frac{3}{(p+3)(p+4)}$$

After two seconds of deep contemplation, you bring forth the following theorem from your vast storehouse of algebraic knowledge:

$$\forall p \quad \frac{3}{(p+3)(p+4)} = \frac{p+1}{p+4} - \frac{p}{p+3}$$

Give students a moment or two to check this, and then proceed.

$$\begin{aligned}
 \sum_{p=1}^{100} \frac{3}{(p+3)(p+4)} &= \sum_{p=1}^{100} \left(\frac{p+1}{p+4} - \frac{p}{p+3} \right) \\
 &= \sum_{p=1}^{100} \frac{p+1}{p+4} - \sum_{p=1}^{100} \frac{p}{p+3} \\
 &= \sum_{p=2}^{101} \frac{(p-1)+1}{(p-1)+4} - \sum_{p=1}^{100} \frac{p}{p+3} \\
 &= \sum_{p=2}^{101} \frac{p}{p+3} - \sum_{p=1}^{100} \frac{p}{p+3}
 \end{aligned}$$

[You can let students suggest the steps from here on.]

$$\begin{aligned}
 &= \left(\sum_{p=2}^{100} \frac{p}{p+3} + \frac{101}{101+3} \right) - \sum_{p=1}^{100} \frac{p}{p+3} \\
 &= \left(\sum_{p=2}^{100} \frac{p}{p+3} + \frac{101}{101+3} \right) - \left(\frac{1}{1+3} + \sum_{p=2}^{100} \frac{p}{p+3} \right) \\
 &= \frac{101}{104} - \frac{1}{4} = \frac{75}{104}
 \end{aligned}$$

So, the pyramid is $\frac{75}{104}$ of an inch high.

The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function. The second part of the paper is devoted to the study of the properties of the function $g(x)$ defined by the equation $g(x) = \int_0^x g(t) dt$. It is shown that $g(x)$ is a constant function.

The third part of the paper is devoted to the study of the properties of the function $h(x)$ defined by the equation $h(x) = \int_0^x h(t) dt$. It is shown that $h(x)$ is a constant function. The fourth part of the paper is devoted to the study of the properties of the function $k(x)$ defined by the equation $k(x) = \int_0^x k(t) dt$. It is shown that $k(x)$ is a constant function.

5. 150 $[= [3(50 + 1) + 7] - [3 \cdot 1 + 7]]$ [Of course, this problem can be worked more readily by simplifying the expression following the Σ -sign. Our purpose here is to give students practice in picking out the ' a_p ' and the ' a_{p+1} '. They are less obvious in the exercises which follow. [Similar remarks hold for Exercises 2, 6, and 7.]
6. 200 $[a_p = 4p + 7, a_{p+1} = 4(p + 1) + 7 = 4p + 11; \text{ students should feel comfortable with obtaining the answer by substituting '50' for 'n' in } '(4n + 11) - (4 \cdot 1 + 7)'.]$

An inductive proof for Theorem 138:

$$(i) \quad \sum_{p=1}^1 (a_{p+1} - a_p) = a_{1+1} - a_1 \quad [\text{by recursive definition}]$$

$$\begin{aligned} (ii) \quad \sum_{p=1}^{q+1} (a_{p+1} - a_p) &= \sum_{p=1}^q (a_{p+1} - a_p) + (a_{q+2} - a_{q+1}) \\ &= (a_{q+1} - a_1) + (a_{q+2} - a_{q+1}) \\ &= a_{q+1+1} - a_1 \end{aligned}$$

For a further discussion of Theorem 138, see TC[8-57]d.

*

Answers for Part A.

$$1. -\frac{50}{51} = \left[\frac{1}{50+1} - \frac{1}{1} \right] \qquad 2. 50 [= (50+1) - 1]$$

$$3. \frac{50}{51} \quad [\text{The } \Sigma\text{-expression needs to be transformed in order to use Theorem 138.}]$$

$$\sum_{p=1}^{50} \left(\frac{1}{p} - \frac{1}{p+1} \right) = \sum_{p=1}^{50} - \left(\frac{1}{p+1} - \frac{1}{p} \right) = - \sum_{p=1}^{50} \left(\frac{1}{p+1} - \frac{1}{p} \right)$$

Students may suggest a corollary to Theorem 138:

$$\forall_n \quad \sum_{p=1}^n (a_p - a_{p+1}) = a_1 - a_{n+1}$$

$$4. \quad 6 \quad [= \sqrt{48+1} - \sqrt{1}]$$

$$7. \quad 100 \quad [a_p = 2p - 1, \quad a_{p+1} = 2(p + 1) - 1 = 2p + 1; \\ (2 \cdot 50 + 1) - (2 \cdot 1 - 1) = 100]$$

$$8. \quad -\frac{100}{101}$$

$$9. \quad \frac{25}{36} \left[a_p = \frac{p}{p+3}, \quad a_{p+1} = \frac{p+1}{p+4} \right]$$

10. $\frac{4}{5}$ [This exercise is loaded. The given expression must be transformed in two ways before Theorem 138 can be applied.

$$\sum_{p=6}^{29} \left(\frac{1}{\sqrt{p-5}} - \frac{1}{\sqrt{p-4}} \right) = \sum_{p=1}^{24} \left(\frac{1}{\sqrt{p}} - \frac{1}{\sqrt{p+1}} \right) = - \sum_{p=1}^{24} \left(\frac{1}{\sqrt{p+1}} - \frac{1}{\sqrt{p}} \right)$$

Also, a student might discover that he can get the right answer if he works as follows:

$$\begin{aligned} \sum_{p=6}^{29} \left(\frac{1}{\sqrt{p-5}} - \frac{1}{\sqrt{p-4}} \right) &= - \sum_{p=6}^{29} \left(\frac{1}{\sqrt{p-4}} - \frac{1}{\sqrt{p-5}} \right) \\ &= - \left(\frac{1}{\sqrt{29-4}} - \frac{1}{\sqrt{6-5}} \right) \end{aligned}$$

He has, in effect, discovered the following generalization of Theorem 138:

$$\forall_j \forall_{k \geq j-1} \sum_{i=j}^k (a_{i+1} - a_i) = a_{k+1} - a_j$$

which is a consequence of Theorems 137 and 138.]

11. 2450 $[a_p = (p-1)(p-2), \quad a_{p+1} = p(p-1);$ if students simplify the expression after the Σ -sign by factoring out the ' $p-1$ ', they will anticipate the work of Exercise 1 of Part C on page 8-55.]

$$12. -2850 \quad [a_p = (p+2)(p+3), a_{p+1} = (p+3)(p+4)]$$

$$13. \frac{n}{2(n+6)} \quad \left[= \frac{n+3}{n+6} - \frac{1+2}{1+5} \right]$$

$$14. \frac{3n}{2(3n+2)} \quad \left[= \frac{1}{3 \cdot 1 - 1} - \frac{1}{3n+2} \right]$$

$$15. \sqrt{n+1} - 1$$

16. $\sqrt{n+1} - \sqrt{m}$ [If you discussed the generalization of Theorem 138 in connection with Exercise 10, students should use that for this problem. If they didn't discover it in connection with that exercise, they will certainly discover it now after working the problem in either of the following ways:

$$\begin{aligned} \sum_{p=m}^n (\sqrt{p+1} - \sqrt{p}) &= \sum_{p=1}^{n-m+1} (\sqrt{p+m} - \sqrt{p+m-1}) \\ &= \sqrt{n-m+1+m} - \sqrt{1+m-1} \\ &= \sqrt{n+1} - \sqrt{m} \end{aligned}$$

or:

$$\begin{aligned} \sum_{p=m}^n (\sqrt{p+1} - \sqrt{p}) &= \sum_{p=1}^n (\sqrt{p+1} - \sqrt{p}) - \\ &\quad \sum_{p=1}^{m-1} (\sqrt{p+1} - \sqrt{p}) \\ &= (\sqrt{n+1} - \sqrt{1}) - (\sqrt{m-1+1} - \sqrt{1}) \\ &= \sqrt{n+1} - \sqrt{m} \quad] \end{aligned}$$

17. $\frac{\sqrt{n+1} - 1}{\sqrt{n+1}}$ [Simplify the expression following the Σ -sign to:

$$\frac{1}{\sqrt{p}} - \frac{1}{\sqrt{p+1}}$$

$$\text{Then, } -\left(\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{1}}\right) = -\left(\frac{1 - \sqrt{n+1}}{\sqrt{n+1}}\right) = \frac{\sqrt{n+1} - 1}{\sqrt{n+1}}.]$$

18. $\sqrt{n+1} - 1$. [Since $p+1 > p$, $\sqrt{p+1} > \sqrt{p}$. So, $\sqrt{p+1} - \sqrt{p} \neq 0$.

$$\begin{aligned}\text{Hence, } \frac{1}{\sqrt{p+1} + \sqrt{p}} &= \frac{1}{\sqrt{p+1} + \sqrt{p}} \cdot \frac{\sqrt{p+1} - \sqrt{p}}{\sqrt{p+1} - \sqrt{p}} \\ &= \frac{\sqrt{p+1} - \sqrt{p}}{p+1 - p} = \sqrt{p+1} - \sqrt{p}.\end{aligned}$$

Now, Exercise 15 can be used.]

*

Answers for Part B.

$$1. \quad q^2 + 2q = \sum_{p=1}^q (2p+1) = 2 \sum_{p=1}^q p + \sum_{p=1}^q 1 = 2 \sum_{p=1}^q p + q$$

$$\text{So, } q^2 + 2q - q = 2 \sum_{p=1}^q p.$$

$$\text{Hence, } \sum_{p=1}^q p = \frac{q^2 + q}{2} = \frac{q(q+1)}{2}.$$

*

Answers for Exercises 2, 3, and ☆4 of Part B are in the COMMENTARY for page 8-55.

*

The procedure illustrated by the exercises of Part B admits of general application once one has the binomial theorem as a uniform source of the requisite algebra theorems. The problem of obtaining summation theorems for sequences of powers is taken up later in this unit [pages 8-202 ff.] as an application of the binomial theorem. The theorems of Part C, below, are also treated [pages 8-205 ff.] in a more systematic manner by making use of the binomial coefficients.

*

Answers for Part C.

$$1. (a) \forall_p p(p - 1) - (p - 1)(p - 2) = (p - 1)[p - (p - 2)] = 2(p - 1)$$

$$(b) \sum_{p=1}^q 2(p - 1) = \sum_{p=1}^q [p(p - 1) - (p - 1)(p - 2)]$$

$$= q(q - 1) - (1 - 1)(1 - 2)$$

$$\text{So, } \sum_{p=1}^q (p - 1) = \frac{q(q - 1)}{2}.$$

$$\begin{aligned}
\text{So, } \sum_{p=1}^q p^3 &= \frac{q^4 + 6 \sum_{p=1}^q p^2 - 4 \sum_{p=1}^q p + \sum_{p=1}^q 1}{4} \\
&= \frac{q^4 + q(q+1)(2q+1) - 2q(q+1) + q}{4} \\
&= \frac{q[(q^3 + 1) + (q+1)(2q+1) - 2(q+1)]}{4}, \\
&= \frac{q(q+1)[(q^2 - q + 1) + (2q+1) - 2]}{4} \\
&= \frac{q(q+1)(q^2 + q)}{4} = \left[\frac{q(q+1)}{2} \right]^2.
\end{aligned}$$

Now, using the algebra theorem:

$$\forall_p p^5 - (p-1)^5 = 5p^4 - 10p^3 + 10p^2 - 5p + 1$$

and Theorem 138, we see that

$$5 \sum_{p=1}^q p^4 = q^5 + 10 \sum_{p=1}^q p^3 - 10 \sum_{p=1}^q p^2 + 5 \sum_{p=1}^q p - \sum_{p=1}^q 1.$$

$$\begin{aligned}
\text{So, } 30 \sum_{p=1}^q p^4 &= 6q^5 + 15[q(q+1)]^2 - 10q(q+1)(2q+1) + 15q(q+1) - 6q \\
&= q[6(q^4 - 1) + 15q(q+1)^2 - 10(q+1)(2q+1) + 15(q+1)] \\
&= q(q+1)[6(q-1)(q^2+1) + 15q(q+1) - 10(2q+1) + 15] \\
&= q(q+1)(6q^3 + 9q^2 + q - 1) \quad [= q(q+1)(2q+1)(3q^2 + 3q - 1)].
\end{aligned}$$

$$\text{Hence, } \sum_{p=1}^q p^4 = \frac{q(q+1)(6q^3 + 9q^2 + q - 1)}{30} \quad [= \frac{q(q+1)(2q+1)(3q^2 + 3q - 1)}{30}].$$

$$\begin{aligned}
 q^3 &= \sum_{p=1}^q (3p^2 - 3p + 1) \\
 &= 3 \sum_{p=1}^q p^2 - 3 \sum_{p=1}^q p + \sum_{p=1}^q 1.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \sum_{p=1}^q p^2 &= \frac{q^3 + \frac{3q(q+1)}{2} - q}{3} \\
 &= \frac{q[q^2 - 1 + \frac{3(q+1)}{2}]}{3} \\
 &= \frac{q(q+1)[2(q-1) + 3]}{6} \\
 &= \frac{q(q+1)(2q+1)}{6}.
 \end{aligned}$$

[The algebraic manipulation is less demanding than in Exercise 2.]

- ★4. [Students should be allowed to discover that they need the summation theorem for the cubes to solve this problem.]

$$\forall_p \ p^4 - (p-1)^4 = 4p^3 - 6p^2 + 4p - 1$$

Since, by Theorem 138, $\sum_{p=1}^q [p^4 - (p-1)^4] = q^4$, it follows that

$$4 \sum_{p=1}^q p^3 - 6 \sum_{p=1}^q p^2 + 4 \sum_{p=1}^q p - \sum_{p=1}^q 1 = q^4.$$

$$2. \quad \forall_p (p + 1)^3 - p^3 = p^3 + 3p^2 + 3p + 1 - p^3 = 3p^2 + 3p + 1$$

Now, since, by Theorem 138,

$$\sum_{p=1}^q [(p + 1)^3 - p^3] = (q + 1)^3 - 1,$$

it follows from the just-proved algebra theorem that

$$\begin{aligned} (q + 1)^3 - 1 &= \sum_{p=1}^q (3p^2 + 3p + 1) \\ &= 3 \sum_{p=1}^q p^2 + 3 \sum_{p=1}^q p + \sum_{p=1}^q 1 = 3 \sum_{p=1}^q p^2 + 3 \frac{q(q + 1)}{2} + q. \end{aligned}$$

$$\begin{aligned} \text{Hence, } \sum_{p=1}^q p^2 &= \frac{1}{3} [(q + 1)^3 - 1 - 3 \frac{q(q + 1)}{2} - q] \\ &= \frac{1}{3} (q + 1) [(q + 1)^2 - \frac{3q}{2} - 1] \\ &= \frac{1}{6} (q + 1) [2(q + 1)^2 - 3q - 2] \\ &= \frac{1}{6} (q + 1) [2q^2 + q] = \frac{q(q + 1)(2q + 1)}{6}. \end{aligned}$$

$$3. \quad \forall_p p^3 - (p - 1)^3 = p^3 - p^3 + 3p^2 - 3p + 1 = 3p^2 - 3p + 1$$

Now, since, by Theorem 138,

$$\sum_{p=1}^q [p^3 - (p - 1)^3] = q^3 - (1 - 1)^3 = q^3,$$

it follows that

$$2. (a) \forall_p p(p-1)(p-2) - (p-1)(p-2)(p-3) = (p-1)(p-2)[p - (p-3)] \\ = 3(p-1)(p-2)$$

$$(b) \sum_{p=1}^q 3(p-1)(p-2) = \sum_{p=1}^q [p(p-1)(p-2) - (p-1)(p-2)(p-3)] \\ = q(q-1)(q-2)$$

So,

$$\sum_{p=1}^q (p-1)(p-2) = \frac{q(q-1)(q-2)}{3}.$$

3. [See Theorems 139c and d on page 8-239. The proofs are quite like those for Exercises 1 and 2.]

*

Answers for Part ☆D.

1. [We trust that the discovery made in Exercise 3 of Part A on page 8-53 and used elsewhere will render the discussion about "remedying the situation" unnecessary.]

$$\sum_{p=1}^q \frac{1}{p(p+1)} = \sum_{p=1}^q \left(\frac{1}{p} - \frac{1}{p+1} \right) = - \sum_{p=1}^q \left(\frac{1}{p+1} - \frac{1}{p} \right) = 1 - \frac{1}{q+1} = \frac{q}{q+1}$$

2. [This theorem has been proved in solving Part D on page 8-13.]

$$\forall_p \frac{1}{2p-1} - \frac{1}{2p+1} = \frac{2p+1 - 2p-1}{(2p-1)(2p+1)} = \frac{2}{(2p-1)(2p+1)}$$

$$\sum_{p=1}^q \frac{1}{(2p-1)(2p+1)} = \frac{1}{2} \sum_{p=1}^q \left(\frac{1}{2p-1} - \frac{1}{2p+1} \right) = \frac{1}{2} \left(\frac{1}{2 \cdot 1 - 1} - \frac{1}{2q+1} \right) = \frac{q}{2q+1}$$

[For the first, $\sum_{i=0}^k a_i - \sum_{i=1}^{k-1} a_i = a_k$; for the second, $\sum_{i=0}^{k-1} (a_{i+1} - a_i) =$

$a_k - a_0$ (cf. Th. 138).] These theorems are analogous to well-known results from the differential and integral calculus:

$$\forall_t D_t \int_0^t f(x) dx = f(t) \quad \text{and:} \quad \forall_t \int_0^t D_x f(x) dx = f(t) - f(0)$$

[The first holds for continuous functions; the second holds for functions which have integrable derivatives.] The same analogy carries over for more special formulas. For example:

$$\forall_t \int_0^t x^2 dx = \frac{t^3}{3} \quad \text{and:} \quad \forall_{k \geq 0} \sum_{i=0}^{k-1} i(i-1) = \frac{k(k-1)(k-2)}{3}$$

The role played in the differential and integral calculus by the power function [the squaring function, the cubing function, etc.] is played in the calculus of finite differences by the descending factorial sequences [the sequences a , b , etc. such that for each $k \geq 0$, $a_k = k(k-1)$, $b_k = k(k-1)(k-2)$, etc.]. [See, for examples, the book on this subject by G. Boole and the book Interpolation by J. F. Steffensen, both published by Chelsea.]

The question about $(\Delta b)_{150}$ forces the student to use the formula, and if he has been observant about how the formula "worked" for the previous two questions, he will proceed as follows:

$$\begin{aligned}(\Delta b)_{150} &= b_{151} - b_{150} = 2(151^2 - 150^2) \\&= 2(151 - 150)(151 + 150) \\&= 2(151 + 150) = 602\end{aligned}$$

He may even be tempted to generalize to:

$$\forall_n (\Delta b)_n = 2(n + 1 + n) = 2(2n + 1)$$

If he does this, he has answered the questions in the next paragraph [last paragraph on page 8-57]. If he doesn't, those questions will lead him to it.

$c_4 = 18$; $c_5 = 22$; $c_{150} = 602$ [Since $c = \Delta b$, these results are not surprising.]

The justification is very easy:

$$\begin{aligned}(\Delta b)_p &= b_{p+1} - b_p = [3 + 2(p+1)^2] - [3 + 2p^2] \\&= 2[(p+1)^2 - p^2] = 2(2p+1)\end{aligned}$$

*

★The relation between summing and differencing becomes more apparent if one considers "sequences" whose domain is the set of nonnegative integers. In this case, if one defines two operators, Σ and Δ , by:

$$\forall_{k \geq 0} (\Sigma a)_k = \sum_{i=0}^{k-1} a_i \quad \text{and:} \quad \forall_{k \geq 0} (\Delta a)_k = a_{k+1} - a_k$$

one obtains without difficulty the theorems:

$$\forall_{k \geq 0} (\Delta(\Sigma a))_k = a_k \quad \text{and:} \quad \forall_{k \geq 0} (\Sigma(\Delta a))_k = a_k - a_0$$

*

The following comments refer to the text on the lower half of page 8-57.

Students should be able to "translate" their intuitive feeling about what a difference-sequence is into an understanding of the formula:

$$\forall p (\Delta a)_p = a_{p+1} - a_p$$

[Pronounce ' Δ ' as 'delta' and ' $(\Delta a)_p$ ' as 'delta a [slight pause] sub p'.]
The questions which follow the formula [questions about the sequence b] are designed to force students into doing some thinking about the formula. Students should form the habit of pondering a new formula until they do understand it. Some students with lots of analytical talent may be able to do this instantly in the present case. But others may need to construct examples, and then "discover" the formula from the examples. For instance, such students might follow the formula and construct and check these consequences:

$$(\Delta a)_1 = a_{1+1} - a_1 = a_2 - a_1 = 18 - 10 = 8 \checkmark$$

$$(\Delta a)_2 = a_3 - a_2 = 28 - 18 = 10 \checkmark$$

$$(\Delta a)_7 = a_8 - a_7 = 108 - 88 = 20$$

we get these
from the list
of terms of a

we check this in
the list of terms
of Δa

If a student does understand the formula, he can handle the questions which follow in a mechanical way. Otherwise, he may need to list the beginning terms of b and of Δb .

$$(\Delta b)_4 = b_5 - b_4 = [3 + 2 \cdot 5^2] - [3 + 2 \cdot 4^2] = 18$$

$$(\Delta b)_5 = b_6 - b_5 = [3 + 2 \cdot 6^2] - [3 + 2 \cdot 5^2] = 22$$

*

An inductive proof for Theorem 138 has been outlined in the COMMENTARY for page 8-53. Like all summation theorems which can be proved by induction, Theorem 138 can also be derived from Theorem 130. According to the latter theorem, in order to prove:

$$\forall_n \sum_{p=1}^n (a_{p+1} - a_p) = a_{n+1} - a_1 \quad [\text{Theorem 138}]$$

it is sufficient to establish two results [" $b_1 = a_1$ " and " $\forall_n b_{n+1} = b_n + a_{n+1}$ "]:

$$a_{1+1} - a_1 = a_{1+1} - a_1$$

and:

$$\forall_n a_{n+1+1} - a_1 = a_{n+1} - a_1 + (a_{n+1+1} - a_{n+1})$$

The first is absolutely trivial and the second is nearly so. So, Theorem 138 follows from Theorem 130.

It is of more interest to note that, with the help of two general theorems on Σ -notation--Theorems 136 and 137--Theorem 130 can be derived from Theorem 138. Thus, Theorems 130 and 138 are merely different ways of saying the same thing.

To derive Theorem 130 from Theorem 138, we begin by assuming that $b_1 = a_1$ and $\forall_n b_{n+1} = b_n + a_{n+1}$. Since, by Theorem 138,

$$\sum_{p=1}^{q-1} (b_{p+1} - b_p) = b_q - a_1, \text{ it follows that } \sum_{p=1}^{q-1} a_{p+1} = b_q - a_1 \text{--that is,}$$

$$\text{that } a_1 + \sum_{p=1}^{q-1} a_{p+1} = b_q. \text{ Now, by Theorem 137, } \sum_{p=1}^{q-1} a_{p+1} = \sum_{p=2}^q a_p,$$

$$\text{and, by Theorem 136, } a_1 + \sum_{p=2}^q a_p = \sum_{p=1}^q a_p. \text{ So, } \sum_{p=1}^q a_p = b_q. \text{ Hence,}$$

$$\forall_n \sum_{p=1}^n a_p = b_n. \text{ Consequently, Theorem 130.}$$

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requirement is necessary, but for $c \neq 0$ it is necessary to require that $-d/c \notin I^+$. Clearly, these conditions are also sufficient. Our restricted quantifier notation does not lend itself readily to expressing these restrictions. [If we did not insist that instances of generalizations be meaningful, we could easily settle the matter by prefixing to (*):

$$\forall_c \forall_d [(d \neq 0 \text{ and } [c \neq 0 \Rightarrow -d/c \notin I^+]) \Rightarrow$$

and adding a closing bracket.] However, it is easy to find various restrictions which, while not necessary, are sufficient and are easy to invoke by restricted quantification. For example, the necessary and sufficient requirements obtained above are evidently satisfied if $c \geq 0$ and $d > 0$. They are also satisfied if $c \leq 0$ and $d < 0$. So, for example, we have the following theorem:

$$(**) \quad \forall_{x \geq 0} \forall_{y > 0} \forall_n \sum_{p=1}^n \frac{1}{[xp - (x - y)](xp + y)} = \frac{n}{y(xn + y)}$$

and a similar theorem with the two inequality signs reversed. [There is a way of stating the "complete" generalization using restricted quantifiers. This is to introduce a definition: $\forall_x S_x = \{z: \forall_{k \leq 0} z \neq xk\}$. Then, replace the first two quantifiers of (**) by ' $\forall_x \forall_y \in S_x$ '.]

Students may discover a summation theorem of a different order of generality:

For any sequence a such that, for each p , $a_p \neq 0$,

$$\forall_n \sum_{p=1}^n \frac{a_{p+1} - a_p}{a_p a_{p+1}} = \frac{a_{n+1} - a_1}{a_1 a_{n+1}}.$$

The basic algebra theorem is simply [ignoring 0-restrictions]:

$$\forall_p \left(\frac{1}{a_p} - \frac{1}{a_{p+1}} \right) = \frac{a_{p+1} - a_p}{a_p a_{p+1}}$$

or, more adequately stated:

$$\forall_{x \neq 0} \forall_{y \neq 0} \left(\frac{1}{x} - \frac{1}{y} \right) = \frac{y - x}{xy}$$

This summation theorem includes the valid cases of (*) [For each p , $a_p = c(p - 1) + d$.].

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summation theorem:

$$\forall_m \forall_n \sum_{p=1}^n \frac{1}{[mp - (m-1)](mp+1)} = \frac{n}{mn+1}$$

5. The summation theorem of Exercise 4 can be generalized to any extent to which the corresponding algebra theorem can be generalized. Ignoring the impossibility of dividing by 0, it is easy to see that

$$\frac{1}{[cp - (c-d)](cp+d)} = \frac{1}{c} \left(\frac{1}{cp - (c-d)} - \frac{1}{cp+d} \right),$$

and, since $cp - (c-d) = c(p-1) + d$, the right side of this equation is of the form:

$$\frac{1}{c} [a_p - a_{p+1}]$$

So, for any choice of real numbers c and d which does not lead to division by 0,

$$(*) \quad \forall_n \sum_{p=1}^n \frac{1}{[cp - (c-d)](cp+d)} = \frac{n}{d(cn+d)},$$

$$\left[\frac{1}{c} \sum_{p=1}^q \left(\frac{1}{c(p-1)+d} - \frac{1}{cp+d} \right) = \frac{1}{c} \left(\frac{1}{d} - \frac{1}{cq+d} \right) = \frac{q}{d(cq+d)} \right]$$

To investigate the necessary restrictions on c and d , we note, first, that although the algebra theorem--and, hence, the proof given for the summation theorem--requires that $c \neq 0$, the summation theorem (*), itself, is correct for $c = 0$. [In this case, it reduces to

$$\forall_n \sum_{p=1}^n 1/d^2 = n/d^2, \text{ which we know to be correct for } d \neq 0.]$$

Evidently, however, we must have $c(p-1) + d \neq 0$ for $p \in I^+$. [This is merely to say that ' $\forall_p a_p = \frac{1}{c(p-1)+d}$ ' is not nonsense.] Since $1 \in I^+$, this leads to the requirement that $d \neq 0$. For $c = 0$ no further

1. The first part of the paper is devoted to a general discussion of the problem.

2. In the second part, we consider the case of a single particle.

3. The third part is devoted to the case of a system of particles.

4. In the fourth part, we consider the case of a continuous medium.

5. The fifth part is devoted to the case of a system of continuous media.

6. In the sixth part, we consider the case of a system of particles and continuous media.

7. The seventh part is devoted to the case of a system of particles and continuous media, with a view to the application of the results to the theory of the propagation of waves in a medium.

8. In the eighth part, we consider the case of a system of particles and continuous media, with a view to the application of the results to the theory of the propagation of waves in a medium.

9. The ninth part is devoted to the case of a system of particles and continuous media, with a view to the application of the results to the theory of the propagation of waves in a medium.

10. In the tenth part, we consider the case of a system of particles and continuous media, with a view to the application of the results to the theory of the propagation of waves in a medium.

11. The eleventh part is devoted to the case of a system of particles and continuous media, with a view to the application of the results to the theory of the propagation of waves in a medium.

3. Completion of the hint suggests that the summation theorem begins:

$$\forall_n \sum_{p=1}^n \frac{1}{(3p-2)(3p+1)} =$$

and Exercises 1 and 2 suggest the algebra theorem:

$$\forall_p \frac{1}{(3p-2)(3p+1)} = \frac{1}{3} \left(\frac{1}{3p-2} - \frac{1}{3p+1} \right)$$

The algebra theorem is easily proved and, with Th. 138 yields:

$$\forall_n \sum_{p=1}^n \frac{1}{(3p-2)(3p+1)} = \frac{n}{3n+1}$$

$$\left[\sum_{p=1}^q \frac{1}{(3p-2)(3p+1)} = \frac{1}{3} \sum_{p=1}^q \left(\frac{1}{3p-2} - \frac{1}{3p+1} \right) = \frac{1}{3} \left(1 - \frac{1}{3q+1} \right) = \frac{q}{3q+1} \right]$$

4. [The theorem in question has been discovered in a different way, and proved by induction, in solving Part ★E on page 8-13. The difference between the two ways of discovering the theorem is significant. In Part ★E, this discovery came about by comparing the summation theorems of Exercises 1 and 2, above. These suggested the theorem of Exercise 3 and, finally, that of Exercise 4. In the present Part ★D, on the other hand, investigation of only the "general terms" of the sequences suggested a summation procedure [use of Theorem 138] which led, simultaneously, to the discovery and proof of the theorems of Exercises 1, 2, and 3 and, in the present exercise, of a generalization of which of these are instances.]

Generalizing on the hint for Exercise 3 we see that if, for each p,

$$a_{p+1} = \frac{1}{mp+1} \text{ then, for each p, } a_{p-1+1} = \frac{1}{m(p-1)+1} = \frac{1}{mp-(m-1)}.$$

Previous experience suggests the algebra theorem:

$$\forall_m \forall_p \frac{1}{[mp-(m-1)](mp+1)} = \frac{1}{m} \left(\frac{1}{mp-(m-1)} - \frac{1}{mp+1} \right)$$

This is easily proved and, as in Exercise 3, yields the desired

Answers for Part A.

1. 14, 16, 18, 20, 22, 24

2. 0, 2, 4, 6, 8, 10, 12

3. 10, 24, 44, 70, 102, 140

4. 10, 24, 44, 70, 102, 140

[Exercises 3 and 4 should be assigned together. In solving them students should become aware that a given difference-sequence is generated by any number of sequences. They should be prompted to examine the sequences s and t more closely and notice that $t - s$ is a constant sequence--the constant sequence 4. This discovery should lead to questions about whether each two sequences whose difference is a constant have the same difference-sequence and about whether each two sequences which have the same difference-sequence differ by a constant. In both cases the answer is 'yes'. In fact, suppose that a and b are sequences whose difference is a constant--that is, suppose that, for each p , $a_{p+1} - b_{p+1} = a_p - b_p$. It follows that, for each p , $a_{p+1} - a_p = b_{p+1} - b_p$ --that is, that $\Delta a = \Delta b$. On the other hand, suppose that a and b are sequences such that, for each p , $a_{p+1} - a_p = b_{p+1} - b_p$. It follows that, for each p , $a_{p+1} - b_{p+1} = a_p - b_p$ and, so, by an easy inductive proof, that, for each p , $a_p - b_p = a_1 - b_1$. So, $a - b$ is the constant sequence whose value is $a_1 - b_1$.

[The results just proved correspond to the calculus theorem that two differentiable functions have the same derivative if and only if their difference is a constant.]

5. 20, 48, 88, 140, 204, 280 [A student on the lookout for short cuts may be able to predict this result by noting that $u = 2s$. Because of this, for each p , $(\Delta u)_p = 2s_{p+1} - 2s_p = 2(s_{p+1} - s_p) = 2(\Delta s)_p$ --that is, $\Delta u = 2\Delta s$.]

6. 12, 28, 50, 78, 112; 16, 22, 28, 34

7. 1, 3, 5, 7, 9, 11, 13, 15, 17; 2, 2, 2, 2, 2, 2, 2, 2

8. 1, 7, 19, 37, 61, 91, 127, 169; 6, 12, 18, 24, 30, 36, 42;
 6, 6, 6, 6, 6

[Exercises 7 and 8 should be assigned together. A student might predict the results in Exercise 7 from his knowledge that, for each n , the sum of the first n positive odd numbers is n^2 . In any event, the two exercises together may suggest that if, starting with a sequence of powers, one forms enough successive difference-sequences, he will arrive eventually at one which is a constant [See page 8-206.]. This is the case, but it is difficult to prove. The difficulty is to be expected since the difference calculus is not well-suited to handle, easily, sequences of powers. Compare, for example, the difficulty of extending Theorem 131, and the ease with which Theorem 132 can be extended.]

Notice that it is customary to abbreviate ' $\Delta(\Delta c)$ ' to ' $\Delta^2 c$ ', ' $\Delta(\Delta(\Delta c))$ ' to ' $\Delta^3 c$ ', etc.

9. [For typographical reasons we list the terms of the sequences in columns, rather than in rows.]

| s | Δs | $\Delta^2 s$ | $\Delta^3 s$ |
|----------------------|----------------|--------------|--------------|
| a | m | | |
| $a + m$ | $m + n$ | n | k |
| $a + 2m + n$ | $m + 2n + k$ | $n + k$ | k |
| $a + 3m + 3n + k$ | $m + 3n + 3k$ | $n + 2k$ | k |
| $a + 4m + 6n + 4k$ | $m + 4n + 6k$ | $n + 3k$ | k |
| $a + 5m + 10n + 10k$ | $m + 5n + 10k$ | $n + 4k$ | |
| $a + 6m + 15n + 20k$ | | | |

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As in the exercises in Part A where, given only the beginning terms of a sequence, students could find only the beginning terms of its difference-sequence, so, in Part B, given only the beginning terms of a difference-sequence, students have no way of determining more than a few terms of a sequence whose difference-sequence it is. So, in both Parts A and B, the instructions are to "find as many terms as you can". Point out to students, if necessary, that even in the case of Exercise B3 the authors may have been considering a sequence x for which $(\Delta x)_9 = 87$.

Students will discover the answers for Part C while doing the exercises of Part B. They may also discover Theorem 140. [If not, they will in the Exploration Exercises.] Incidentally, Theorem 140 is scarcely more than a reformulation of Theorem 138 which, as is pointed out on TC[8-57]d, is, itself, a reformulation of Theorem 130. [You, yourself, may recognize in Theorem 138 a generalization of the familiar formula ' $a_n = a_1 + (n-1)d$ ' of the theory of arithmetic progressions.]

Answers for Part B.

[Students should see the need for choosing a first term-- a_1 , for Exercise 1--in order to get started. The brighter--as well as the less decisive ones--may even state their answers in the form given here.]

1. $a_1, a_1 + 3, a_1 + 8, a_1 + 16, a_1 + 28, a_1 + 45, a_1 + 68, a_1 + 98, a_1 + 136$
2. $v_1, v_1 + 8, v_1 + 17, v_1 + 27, v_1 + 38, v_1 + 50, v_1 + 63, v_1 + 77, v_1 + 92$
3. $x_1, x_1 + 3, x_1 + 6, x_1 + 9, x_1 + 12, x_1 + 15, x_1 + 18, x_1 + 21, x_1 + 24$
4. $y_1, y_1 - 2, y_1 - 4, y_1 - 6, y_1 - 8, y_1 - 10, y_1 - 12, y_1 - 14, y_1 - 16$
5. $\Delta z: (\Delta z)_1, (\Delta z)_1 + 5, (\Delta z)_1 + 9, (\Delta z)_1 + 12, (\Delta z)_1 + 14, (\Delta z)_1 + 15,$
 $(\Delta z)_1 + 15, (\Delta z)_1 + 14$
 $z: z_1, z_1 + (\Delta z)_1, z_1 + 2(\Delta z)_1 + 5, z_1 + 3(\Delta z)_1 + 14, z_1 + 4(\Delta z)_1 + 26,$
 $z_1 + 5(\Delta z)_1 + 40, z_1 + 6(\Delta z)_1 + 55, z_1 + 7(\Delta z)_1 + 70, z_1 + 8(\Delta z)_1 + 84$

Answers for Part C.

Infinitely many; of these, just one has a given first term.

Answers for the Exploration Exercises are on TC[8-60].

Answers for Exploration Exercises [which begin on page 8-59]

1. 1229 [Students whose have not yet discovered Theorem 140 should

$$\text{do so here. } a_{50} = 4 + 1 + 2 + \dots = 4 + \sum_{p=1}^{49} p = 4 + \frac{49 \cdot 50}{2}]$$

$$2. 1232 [a_{50} = 7 + \sum_{p=1}^{49} p] \quad 3. 148 [b_{50} = 1 + \sum_{p=1}^{49} 3 = 1 + 3 \cdot 49]$$

$$4. 106 [b_{50} = 8 + \sum_{p=1}^{49} 2] \quad 5. 1178 [t_{50} = 2 + \sum_{p=1}^{49} (p-1) = 2 + \frac{49 \cdot 48}{2}]$$

$$6. 36855 [s_{50} = 7 + \sum_{p=1}^{49} (p-1)(p-2) = 7 + \frac{49 \cdot 48 \cdot 47}{3}]$$

*

Proof of Theorem 140 [Except for the instance for $n = 1$, Theorem 140 is a restatement of Theorem 138.]: For any positive integer q , $q = 1$ or $q - 1 \in I^+$. If $q = 1$ then, by the recursive definition for Σ -notation,

$$\sum_{p=1}^{q-1} (\Delta a)_p = 0 = a_1 - a_1. \quad \text{If } q - 1 \in I^+ \text{ then, by Theorem 138, } \sum_{p=1}^{q-1} (\Delta a)_p$$

$$= a_q - a_1. \quad \text{So, in any case, } \sum_{p=1}^{q-1} (\Delta a)_p = a_q - a_1 \text{---that is, } a_q = a_1 + \sum_{p=1}^{q-1} (\Delta a)_p.$$

1. The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \int_0^x \frac{1}{1+t^2} dt, \quad (1)$$

where x is a real number. It is well known that this function is increasing and concave down on the interval $(-\infty, \infty)$.

2. In the second part, we consider the function $g(x)$ defined by the equation

$$g(x) = \int_0^x \frac{t}{1+t^2} dt, \quad (2)$$

where x is a real number. It is well known that this function is increasing and concave up on the interval $(-\infty, \infty)$.

3. In the third part, we consider the function $h(x)$ defined by the equation

$$h(x) = \int_0^x \frac{t^2}{1+t^2} dt, \quad (3)$$

where x is a real number. It is well known that this function is increasing and concave down on the interval $(-\infty, \infty)$.

4. In the fourth part, we consider the function $k(x)$ defined by the equation

$$k(x) = \int_0^x \frac{t^3}{1+t^2} dt, \quad (4)$$

where x is a real number. It is well known that this function is increasing and concave up on the interval $(-\infty, \infty)$.

5. In the fifth part, we consider the function $l(x)$ defined by the equation

$$l(x) = \int_0^x \frac{t^4}{1+t^2} dt, \quad (5)$$

where x is a real number. It is well known that this function is increasing and concave down on the interval $(-\infty, \infty)$.

6. In the sixth part, we consider the function $m(x)$ defined by the equation

$$m(x) = \int_0^x \frac{t^5}{1+t^2} dt, \quad (6)$$

where x is a real number. It is well known that this function is increasing and concave up on the interval $(-\infty, \infty)$.

In the second illustration of the use of Theorem 140 to prove summation theorems, we make use of Theorem 139. This is more efficient in such problems than is Theorem 131 and leads to an algorithm which should be apparent by the end of the illustration. This algorithm is developed more explicitly later in the unit [see Theorem 180].

Notice that the application of Theorem 139a to obtain line 4 on page 8-62 makes use of the instance for $n = q - 1$.

*

Answers for Part A [bottom of page 8-62].

1. 300 [Since $\Delta(\Delta a) = \Delta^2 a$, it follows from Theorem 140 that

$$\begin{aligned}(\Delta a)_q &= (\Delta a)_1 + \sum_{p=1}^{q-1} (\Delta^2 a)_p \\&= 5 + \sum_{p=1}^{q-1} 7 \\&= 5 + 7(q - 1).\end{aligned}$$

Hence, by Theorem 140,

$$\begin{aligned}a_q &= a_1 + \sum_{p=1}^{q-1} (\Delta a)_p \\&= 3 + \sum_{p=1}^{q-1} [5 + 7(p - 1)] \\&= 3 + 5(q - 1) + 7 \frac{(q - 1)(q - 2)}{2}\end{aligned}$$

Consequently, $a_{10} = 3 + 5 \cdot 9 + 7 \frac{9 \cdot 8}{2} = 300.$]

Answers to the remaining exercises in Part A are given in the COMMENTARY for pages 8-63 and 8-64.

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16.

$$\begin{aligned}
 (b) \quad \forall_n \sum_{p=1}^n p^6 &= n + 63 \frac{n(n-1)}{2} + 602 \frac{n(n-1)(n-2)}{2 \cdot 3} \\
 &+ 2100 \frac{n(n-1)(n-2)(n-3)}{2 \cdot 3 \cdot 4} \\
 &+ 3360 \frac{n(n-1)(n-2)(n-3)(n-4)}{2 \cdot 3 \cdot 4 \cdot 5} \\
 &+ 2520 \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \\
 &+ 720 \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}
 \end{aligned}$$

$$\left[\text{Another formula is: } \sum_{p=1}^n p^6 = \left(\frac{3n^4 + 6n^3 - 3n + 1}{7} \right) \left(\frac{n(n+1)(2n+1)}{6} \right) \right]$$

5. (a) Since the Δ^3 -sequence is the first constant difference-sequence for the sequence of cubes, any sequence obtained by adding a constant to the sequence of cubes will do. Also, any sequence obtained by multiplying the sequence of cubes by a constant will do. We hope that a student will generalize on his experiences in the preceding exercises and come out with something like the following:

a is a sequence whose first constant difference-sequence is the third if and only if there are real numbers x_1, x_2, x_3 , and $x_4 \neq 0$ such that

$$\forall_n a_n = x_1 + x_2(n-1) + x_3 \frac{(n-1)(n-2)}{2} + x_4 \frac{(n-1)(n-2)(n-3)}{2 \cdot 3}.$$

Of course, $x_1 = a_1$, $x_2 = (\Delta a)_1$, $x_3 = (\Delta^2 a)_1$, and $\forall_p (\Delta^3 a)_p = x_4$.

$$\begin{aligned}
 (b) \quad \sum_{p=1}^n a_p &= \sum_{p=1}^n \left[x_1 + x_2(p-1) + x_3 \frac{(p-1)(p-2)}{2} + x_4 \frac{(p-1)(p-2)(p-3)}{2 \cdot 3} \right] \\
 &= x_1 n + x_2 \frac{n(n-1)}{2} + x_3 \frac{n(n-1)(n-2)}{2 \cdot 3} + x_4 \frac{n(n-1)(n-2)(n-3)}{2 \cdot 3 \cdot 4}
 \end{aligned}$$

Answer for Exercise 6 of Part C is in the COMMENTARY for page 8-65.

4. (a) Yes, the first of the successive difference-sequences which is a constant is $\Delta^6 a$. [We trust that the student will determine this by experimenting. Here is a table he might prepare [it contains more entries than he should need to make his conjecture]:

| a | Δa | $\Delta^2 a$ | $\Delta^3 a$ | $\Delta^4 a$ | $\Delta^5 a$ | $\Delta^6 a$ |
|---------|------------|--------------|--------------|--------------|--------------|--------------|
| 1 | | | | | | |
| | 63 | | | | | |
| 64 | | 602 | | | | |
| | 665 | | 2100 | | | |
| 729 | | 2702 | | 3360 | | |
| | 3367 | | 5460 | | 2520 | |
| 4096 | | 8162 | | 5880 | | 720 |
| | 11529 | | 11340 | | 3240 | |
| 15625 | | 19502 | | 9120 | | 720 |
| | 31031 | | 20460 | | 3960 | |
| 46656 | | 39962 | | 13080 | | 720 |
| | 70993 | | 33540 | | 4680 | |
| 117649 | | 73502 | | 17760 | | 720 |
| | 144495 | | 51300 | | 5400 | |
| 262144 | | 124802 | | 23160 | | |
| | 269297 | | 74460 | | | |
| 531441 | | 199262 | | | | |
| | 468559 | | | | | |
| 1000000 | | | | | | |

This guess can be proved by using the technique illustrated in Exercise 2. It involves some heavy algebra.]

1. $\sum_{i=1}^n \frac{1}{i^2} = \frac{\pi^2}{6}$ (Euler's result)

2. $\sum_{i=1}^n \frac{1}{i^4} = \frac{\pi^4}{96}$ (Euler's result)

3.

4. $\sum_{i=1}^n \frac{1}{i^6} = \frac{\pi^6}{945}$ (Euler's result)

5. $\sum_{i=1}^n \frac{1}{i^8} = \frac{\pi^8}{75120}$ (Euler's result)

6.

7. $\sum_{i=1}^n \frac{1}{i^{10}} = \frac{\pi^{10}}{93555360}$ (Euler's result)

8.

9.

10. $\sum_{i=1}^n \frac{1}{i^{12}} = \frac{\pi^{12}}{693777600}$ (Euler's result)

11. $\sum_{i=1}^n \frac{1}{i^{14}} = \frac{\pi^{14}}{1352876800}$ (Euler's result)

12. $\sum_{i=1}^n \frac{1}{i^{16}} = \frac{\pi^{16}}{3517030400}$ (Euler's result)

13. $\sum_{i=1}^n \frac{1}{i^{18}} = \frac{\pi^{18}}{1201353600}$ (Euler's result)

14. $\sum_{i=1}^n \frac{1}{i^{20}} = \frac{\pi^{20}}{637559400}$ (Euler's result)

15. $\sum_{i=1}^n \frac{1}{i^{22}} = \frac{\pi^{22}}{238145600}$ (Euler's result)

16. $\sum_{i=1}^n \frac{1}{i^{24}} = \frac{\pi^{24}}{87518400}$ (Euler's result)

17. $\sum_{i=1}^n \frac{1}{i^{26}} = \frac{\pi^{26}}{31364160}$ (Euler's result)

18. $\sum_{i=1}^n \frac{1}{i^{28}} = \frac{\pi^{28}}{10828800}$ (Euler's result)

19. $\sum_{i=1}^n \frac{1}{i^{30}} = \frac{\pi^{30}}{3709376}$ (Euler's result)

20. $\sum_{i=1}^n \frac{1}{i^{32}} = \frac{\pi^{32}}{1294272}$ (Euler's result)

21. $\sum_{i=1}^n \frac{1}{i^{34}} = \frac{\pi^{34}}{429456}$ (Euler's result)

22. $\sum_{i=1}^n \frac{1}{i^{36}} = \frac{\pi^{36}}{141296}$ (Euler's result)

$$\sum_{p=1}^n a_p = \sum_{p=1}^n [7 + d(p-1)] = 7n + d \frac{n(n-1)}{2} \quad \left[= \frac{n}{2} [2a_1 + (n-1)d] \right]$$

However, more attention is paid to such sequences on pages 8-66 ff.]

*

Answers for Part B.

From the results obtained in Exercise 5 of Part A it follows that

$$a_{78} = 7 + 11(78 - 1) = 854 \quad \text{and} \quad \sum_{p=1}^{100} a_p = 7 \cdot 100 + 11 \frac{100 \cdot 99}{2} = 55150.$$

*

Answers for Part C.

1. If $(\Delta^3 a)_4 = 6$ then $(\Delta^2 a)_5 = 30 + 6 = 36$, $(\Delta a)_6 = 91 + 36 = 127$, and $a_7 = 216 + 127 = 343 = 7^3$.

$$\begin{aligned} 2. \quad a_p &= 1 + 7(p-1) + 12 \frac{(p-1)(p-2)}{2} + 6 \frac{(p-1)(p-2)(p-3)}{2 \cdot 3} \\ &= 1 + 7(p-1) + 6(p-1)(p-2) + (p-1)(p-2)(p-3) \\ &= 1 + 7(p-1) + (p-1)(p-2)[6 + (p-3)] \\ &= 1 + (p-1)[7 + (p-2)(p+3)] \\ &= 1 + (p-1)(p^2 + p + 1) = 1 + (p^3 - 1) = p^3 \end{aligned}$$

$$3. \quad 12 \frac{n(n-1)(n-2)}{2 \cdot 3} + 6 \frac{n(n-1)(n-2)(n-3)}{2 \cdot 3 \cdot 4}$$

$$\left[\sum_{p=1}^q a_p = \sum_{p=1}^q \left[1 + 7(p-1) + 12 \frac{(p-1)(p-2)}{2} + 6 \frac{(p-1)(p-2)(p-3)}{2 \cdot 3} \right] \right]$$

$$= q + 7 \frac{q(q-1)}{2} + 12 \frac{q(q-1)(q-2)}{2 \cdot 3} + 6 \frac{q(q-1)(q-2)(q-3)}{2 \cdot 3 \cdot 4}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = 1$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = 1$$

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$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = 1$$

$$\sum_{p=1}^{q-1} (p-1)p = \sum_{p=0}^{q-2} p(p+1) = \sum_{p=1}^{q-2} p(p+1) = \frac{(q-2)(q-1)q}{3}$$

$$\sum_{p=1}^{q-1} (p-1)p = \sum_{p=2}^q (p-2)(p-1) = \sum_{p=1}^q (p-1)(p-2) = \frac{q(q-1)(q-2)}{3}$$

The first makes use of Theorems 133, 134, and 131; the second and third make use of Theorems 137 and 136, and of Theorem 132 or Theorem 139. In any case, the result is that

$$a_q = 1 + 5(q-1) + \frac{q(q-1)(q-2)}{2 \cdot 3}$$

Hence,

$$\begin{aligned} \sum_{p=1}^n a_p &= \sum_{p=1}^n \left[1 + 5(p-1) + \frac{p(p-1)(p-2)}{2 \cdot 3} \right] \\ &= n + 5 \frac{n(n-1)}{2} + \frac{1}{6} \sum_{p=1}^n p(p-1)(p-2). \end{aligned}$$

Again, there are various possibilities, all of which lead to the result:

$$\sum_{p=1}^n a_p = n + 5 \frac{n(n-1)}{2} + \frac{(n+1)n(n-1)(n-2)}{2 \cdot 3 \cdot 4}$$

It is easy to see that this is consistent with the answer given above.

$$5. \quad 7 + d(n-1); \quad 7n + d \frac{n(n-1)}{2}$$

[This exercise takes care of arithmetic progressions.]

$$a_q = a_1 + \sum_{p=1}^{q-1} (\Delta a)_p = 7 + \sum_{p=1}^{q-1} d = 7 + d(q-1) \quad [= a_1 + (q-1)d]$$

However, a student who fails to make this observation can still do the exercise by using Theorem 140 and appropriate summation theorems. Here's how he would begin.

$$(\Delta a)_q = (\Delta a)_1 + \sum_{p=1}^{q-1} (\Delta^2 a)_p$$

$$= 5 + \sum_{p=1}^{q-1} p$$

$$= 5 + \frac{(q-1)q}{2}$$

$$a_q = a_1 + \sum_{p=1}^{q-1} (\Delta a)_p$$

$$= 1 + \sum_{p=1}^{q-1} \left[5 + \frac{(p-1)p}{2} \right]$$

$$= 1 + 5(q-1) + \frac{1}{2} \sum_{p=1}^{q-1} (p-1)p$$

The student now has the problem of evaluating $\sum_{p=1}^{q-1} (p-1)p$.

Here are three ways:

$$\begin{aligned} \sum_{p=1}^{q-1} (p-1)p &= \sum_{p=1}^{q-1} (p^2 - p) = \sum_{p=1}^{q-1} p^2 - \sum_{p=1}^{q-1} p \\ &= \frac{(q-1)q(2q-1)}{6} - \frac{(q-1)q}{2} \\ &= \frac{(q-1)q(q-2)}{3} \end{aligned}$$

2. 1685 [As in Exercise 1, we have:

$$a_q = 5 + 7(q - 1) + 11 \frac{(q - 1)(q - 2)}{2}$$

$$\begin{aligned} \text{So, } \sum_{p=1}^n a_p &= \sum_{p=1}^n \left[5 + 7(p - 1) + 11 \frac{(p - 1)(p - 2)}{2} \right] \\ &= 5n + 7 \frac{n(n - 1)}{2} + 11 \frac{n(n - 1)(n - 2)}{2 \cdot 3}. \end{aligned}$$

$$\begin{aligned} \text{Therefore, } \sum_{p=1}^{10} a_p &= 5 \cdot 10 + 7 \cdot \frac{10 \cdot 9}{2} + 11 \cdot \frac{10 \cdot 9 \cdot 8}{2 \cdot 3} \\ &= 50 + 315 + 1320. \end{aligned} \quad]$$

3. 17450 [As in the second example on pages 8-61 and 8-62, we have:

$$a_q = -11 + 7(q - 1) + 5 \frac{(q - 1)(q - 2)}{2} + 17 \frac{(q - 1)(q - 2)(q - 3)}{2 \cdot 3}$$

$$\begin{aligned} \text{So, } a_{20} &= -11 + 7 \cdot 19 + 5 \cdot \frac{19 \cdot 18}{2} + 17 \cdot \frac{19 \cdot 18 \cdot 17}{2 \cdot 3} \\ &= -11 + 133 + 855 + 16473. \end{aligned} \quad]$$

$$4. \quad n + 5 \frac{n(n - 1)}{2} + 1 \frac{n(n - 1)(n - 2)}{2 \cdot 3} + 1 \frac{n(n - 1)(n - 2)(n - 3)}{2 \cdot 3 \cdot 4}$$

This exercise requires nothing more than an additional step in the procedure for Exercise 3 if one notes that ,

$$\forall_p (\Delta^3 a)_p = (\Delta^2 a)_{p+1} - (\Delta^2 a)_p = p + 1 - p = 1,$$

and that $(\Delta^2 a)_1 = 1$. Then, since

$$a_q = 1 + 5(q - 1) + 1 \frac{(q - 1)(q - 2)}{2} + 1 \frac{(q - 1)(q - 2)(q - 3)}{2 \cdot 3},$$

$$\sum_{p=1}^n a_p = n + 5 \frac{n(n - 1)}{2} + 1 \frac{n(n - 1)(n - 2)}{2 \cdot 3} + 1 \frac{n(n - 1)(n - 2)(n - 3)}{2 \cdot 3 \cdot 4}.$$

3.

| j | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|--------------|---|------|------|------|------|-----|------|
| given data | 0 | 0.46 | 1.24 | 2.35 | 3.77 | 5.5 | 7.55 |
| formula data | 0 | 0.46 | 1.24 | 2.34 | 3.76 | 5.5 | 7.56 |

4. $d_{10} = 19$: $d_{10/3} = 0.16 \cdot \frac{100}{9} + 0.3 \cdot \frac{10}{3} \doteq 2.78$

[The bracketed question is meant to bring up a little discussion of the problems which arise in extrapolating and interpolating from empirical data. Perhaps at the end of the 6/10 second the body landed in water and slowed down. Perhaps during each tenth-second the body fell with constant velocity, increasing its speed suddenly at the end of each tenth-second.]

5. $s_t [= d_{10t}] = 16t^2 + 3t$

[illegible]

1. *Chlorophyll a* (Chl *a*)
 2. *Chlorophyll b* (Chl *b*)
 3. *Chlorophyll c* (Chl *c*)
 4. *Chlorophyll d* (Chl *d*)
 5. *Chlorophyll e* (Chl *e*)
 6. *Chlorophyll f* (Chl *f*)
 7. *Chlorophyll g* (Chl *g*)
 8. *Chlorophyll h* (Chl *h*)
 9. *Chlorophyll i* (Chl *i*)
 10. *Chlorophyll j* (Chl *j*)
 11. *Chlorophyll k* (Chl *k*)
 12. *Chlorophyll l* (Chl *l*)
 13. *Chlorophyll m* (Chl *m*)
 14. *Chlorophyll n* (Chl *n*)
 15. *Chlorophyll o* (Chl *o*)
 16. *Chlorophyll p* (Chl *p*)
 17. *Chlorophyll q* (Chl *q*)
 18. *Chlorophyll r* (Chl *r*)
 19. *Chlorophyll s* (Chl *s*)
 20. *Chlorophyll t* (Chl *t*)
 21. *Chlorophyll u* (Chl *u*)
 22. *Chlorophyll v* (Chl *v*)
 23. *Chlorophyll w* (Chl *w*)
 24. *Chlorophyll x* (Chl *x*)
 25. *Chlorophyll y* (Chl *y*)
 26. *Chlorophyll z* (Chl *z*)
 27. *Chlorophyll aa* (Chl *aa*)
 28. *Chlorophyll ab* (Chl *ab*)
 29. *Chlorophyll ac* (Chl *ac*)
 30. *Chlorophyll ad* (Chl *ad*)
 31. *Chlorophyll ae* (Chl *ae*)
 32. *Chlorophyll af* (Chl *af*)
 33. *Chlorophyll ag* (Chl *ag*)
 34. *Chlorophyll ah* (Chl *ah*)
 35. *Chlorophyll ai* (Chl *ai*)
 36. *Chlorophyll aj* (Chl *aj*)
 37. *Chlorophyll ak* (Chl *ak*)
 38. *Chlorophyll al* (Chl *al*)
 39. *Chlorophyll am* (Chl *am*)
 40. *Chlorophyll an* (Chl *an*)
 41. *Chlorophyll ao* (Chl *ao*)
 42. *Chlorophyll ap* (Chl *ap*)
 43. *Chlorophyll aq* (Chl *aq*)
 44. *Chlorophyll ar* (Chl *ar*)
 45. *Chlorophyll as* (Chl *as*)
 46. *Chlorophyll at* (Chl *at*)
 47. *Chlorophyll au* (Chl *au*)
 48. *Chlorophyll av* (Chl *av*)
 49. *Chlorophyll aw* (Chl *aw*)
 50. *Chlorophyll ax* (Chl *ax*)
 51. *Chlorophyll ay* (Chl *ay*)
 52. *Chlorophyll az* (Chl *az*)
 53. *Chlorophyll aza* (Chl *aza*)
 54. *Chlorophyll abz* (Chl *abz*)
 55. *Chlorophyll acz* (Chl *acz*)
 56. *Chlorophyll adz* (Chl *adz*)
 57. *Chlorophyll aez* (Chl *aez*)
 58. *Chlorophyll afz* (Chl *afz*)
 59. *Chlorophyll agz* (Chl *agz*)
 60. *Chlorophyll ahz* (Chl *ahz*)
 61. *Chlorophyll aiz* (Chl *aiz*)
 62. *Chlorophyll ajz* (Chl *ajz*)
 63. *Chlorophyll akz* (Chl *akz*)
 64. *Chlorophyll alz* (Chl *alz*)
 65. *Chlorophyll amz* (Chl *amz*)
 66. *Chlorophyll anz* (Chl *anz*)
 67. *Chlorophyll aoz* (Chl *aoz*)
 68. *Chlorophyll apz* (Chl *apz*)
 69. *Chlorophyll aqz* (Chl *aqz*)
 70. *Chlorophyll arz* (Chl *arz*)
 71. *Chlorophyll asz* (Chl *asz*)
 72. *Chlorophyll atz* (Chl *atz*)
 73. *Chlorophyll auz* (Chl *auz*)
 74. *Chlorophyll avz* (Chl *avz*)
 75. *Chlorophyll awz* (Chl *awz*)
 76. *Chlorophyll axz* (Chl *axz*)
 77. *Chlorophyll ayz* (Chl *ayz*)
 78. *Chlorophyll azz* (Chl *azz*)
 79. *Chlorophyll azaa* (Chl *aza*)
 80. *Chlorophyll abz* (Chl *abz*)
 81. *Chlorophyll acz* (Chl *acz*)
 82. *Chlorophyll adz* (Chl *adz*)
 83. *Chlorophyll aez* (Chl *aez*)
 84. *Chlorophyll afz* (Chl *afz*)
 85. *Chlorophyll agz* (Chl *agz*)
 86. *Chlorophyll ahz* (Chl *ahz*)
 87. *Chlorophyll aiz* (Chl *aiz*)
 88. *Chlorophyll ajz* (Chl *ajz*)
 89. *Chlorophyll akz* (Chl *akz*)
 90. *Chlorophyll alz* (Chl *alz*)
 91. *Chlorophyll amz* (Chl *amz*)
 92. *Chlorophyll anz* (Chl *anz*)
 93. *Chlorophyll aoz* (Chl *aoz*)
 94. *Chlorophyll apz* (Chl *apz*)
 95. *Chlorophyll aqz* (Chl *aqz*)
 96. *Chlorophyll arz* (Chl *arz*)
 97. *Chlorophyll asz* (Chl *asz*)
 98. *Chlorophyll atz* (Chl *atz*)
 99. *Chlorophyll auz* (Chl *auz*)
 100. *Chlorophyll avz* (Chl *avz*)
 101. *Chlorophyll awz* (Chl *awz*)
 102. *Chlorophyll axz* (Chl *axz*)
 103. *Chlorophyll ayz* (Chl *ayz*)
 104. *Chlorophyll azz* (Chl *azz*)
 105. *Chlorophyll azaa* (Chl *aza*)
 106. *Chlorophyll abz* (Chl *abz*)
 107. *Chlorophyll acz* (Chl *acz*)
 108. *Chlorophyll adz* (Chl *adz*)
 109. *Chlorophyll aez* (Chl *aez*)
 110. *Chlorophyll afz* (Chl *afz*)
 111. *Chlorophyll agz* (Chl *agz*)
 112. *Chlorophyll ahz* (Chl *ahz*)
 113. *Chlorophyll aiz* (Chl *aiz*)
 114. *Chlorophyll ajz* (Chl *ajz*)
 115. *Chlorophyll akz* (Chl *akz*)
 116. *Chlorophyll alz* (Chl *alz*)
 117. *Chlorophyll amz* (Chl *amz*)
 118. *Chlorophyll anz* (Chl *anz*)
 119. *Chlorophyll aoz* (Chl *aoz*)
 120. *Chlorophyll apz* (Chl *apz*)
 121. *Chlorophyll aqz* (Chl *aqz*)
 122. *Chlorophyll arz* (Chl *arz*)
 123. *Chlorophyll asz* (Chl *asz*)
 124. *Chlorophyll atz* (Chl *atz*)
 125. *Chlorophyll auz* (Chl *auz*)
 126. *Chlorophyll avz* (Chl *avz*)
 127. *Chlorophyll awz* (Chl *awz*)
 128. *Chlorophyll axz* (Chl *axz*)
 129. *Chlorophyll ayz* (Chl *ayz*)
 130. *Chlorophyll azz* (Chl *azz*)
 131. *Chlorophyll azaa* (Chl *aza*)
 132. *Chlorophyll abz* (Chl *abz*)
 133.

6. [This exercise sums things up.]

(a) $n - 1$

(b) $s + r(n - 1); \quad t + s(n - 1) + r \frac{(n - 1)(n - 2)}{2}$

(c) $w + t(n - 1) + s \frac{(n - 1)(n - 2)}{2} + r \frac{(n - 1)(n - 2)(n - 3)}{2 \cdot 3}$

*

Answers for Part D.

1.

| j | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|------------------|------|------|------|------|------|------|------|
| d_j | 0 | 0.46 | 1.24 | 2.35 | 3.77 | 5.5 | 7.55 |
| $(\Delta d)_j$ | 0.46 | 0.78 | 1.11 | 1.42 | 1.73 | 2.05 | |
| $(\Delta^2 d)_j$ | 0.32 | 0.33 | 0.31 | 0.31 | 0.32 | | |

$$2. \quad (\Delta d)_j = (\Delta d)_0 + \sum_{k=0}^{j-1} (\Delta^2 d)_k = 0.46 + \sum_{k=0}^{j-1} 0.32$$

$$= 0.46 + 0.32j \quad [0 \leq j \leq 5]$$

$$d_j = d_0 + \sum_{k=0}^{j-1} (0.46 + 0.32k) = 0 + \sum_{k=0}^{j-1} 0.46 + \sum_{k=1}^j 0.32(k-1)$$

$$= 0.46j + 0.32 \sum_{k=1}^j (k-1) = 0.46j + 0.16j(j-1) \quad [0 \leq j \leq 6]$$

$$\text{Formula: } d_j = 0.16j^2 + 0.3j \quad [0 \leq j \leq 6]$$

Much of the material in the ten pages on arithmetic progressions is included to prepare students for conventional test questions. As should be clear, the theory of arithmetic progressions is an almost trivial application of the preceding work on difference-sequences.

*

For each of the given lists, (1) - (8), there is an infinite number of sequences each of which has the listed numbers, in the given order, as its first seven terms. At most one of these sequences is an arithmetic progression.

Arithmetic progressions: (1), 2, 19; (2), 4, 25;
(5), 2π , 15π ; (6), -7, -41

*

Answers for Part A.

1. -2, -1, 0, 1, 2, 3, 4, ...
2. 3, 24, 45, 66, 87, 108, ...
3. 9, 10, 11, 12, 13, 14, 15, ...
4. -8, $-3\frac{5}{4}$, $\frac{1}{3}$, $4\frac{1}{2}$, $8\frac{2}{3}$, $12\frac{5}{6}$, 17, ...
5. 4, 8, 12, 16, 20, 24, ...
6. -5, -10, -15, -20, -25, -30, -35, ...
7. 6, $6\frac{1}{6}$, $6\frac{1}{3}$, $6\frac{1}{2}$, $6\frac{2}{3}$, $6\frac{5}{6}$, 7, ...
8. $\frac{1}{3}$, $\frac{5}{6}$, $\frac{4}{3}$, $\frac{11}{6}$, $\frac{7}{3}$, $\frac{17}{6}$, ...
9. impossible [There is no d such that $3 + 2d = 7$ and $3 + 4d = 9$.]
10. -3 , $-3+2\sqrt{2}$, $-3+4\sqrt{2}$, $-3+6\sqrt{2}$, $-3+8\sqrt{2}$, $-3+10\sqrt{2}$, ...
11. $7-4x$, $7-3x$, $7-2x$, $7-x$, 7, $7+x$, $7+2x$, ...

Answers for the exercises of Part B are given in the COMMENTARY for pages 8-67 and 8-68.

$$\begin{aligned}
 2. \quad \sum_{p=1}^q a_p &= \frac{q}{2} [2a_1 + (q-1)d] && [\text{Ex. 1}] \\
 &= \frac{q}{2} [a_1 + [a_1 + (q-1)d]] \\
 &= \frac{q}{2} [a_1 + a_q] && \left. \vphantom{\sum_{p=1}^q a_p} \right\} \text{Theorem 141a}
 \end{aligned}$$

*

Answers for Part E.

- | | | | | |
|--|-------------------------------|---------|----------|--------|
| 1. 1290 | 2. 5400 | 3. -3.5 | 4. 462.5 | 5. 108 |
| 6. 380 | 7. 1000000 [Recall page 8-1.] | | | |
| 8. $14(81 + 2\sqrt{2})$ | 9. -2501499 | 10. 1 | | |
| 11. 1220 $\left[= \frac{20}{2} (2[7 + 4 \cdot 4] + 19 \cdot 4) \right]$ | | | | |
| 12. -17232 $\left[= \frac{48}{2} (2[64 + 47 \cdot -6] + 47 \cdot -6) \right]$ | | | | |
| 13. 500500 $\left[= \frac{1000}{2} (1 + 1000) \right]$ | | | | |

Answers for Part B [which begins on page 8-66].

1. 114 [= 4 + 11 · 10]

2. -37 [= 3 + 10 · -4]

$$3. a_n = a_1 + \sum_{p=1}^{n-1} d = a_1 + d(n-1)$$

$$4. a_m - a_n = [a_1 + d(m-1)] - [a_1 + d(n-1)] \\ = d(m-1-n+1) = d(m-n)$$

$$\frac{a_m - a_n}{m - n} = d, \quad [m \neq n]$$

*

Answers for Part C.

1. 173

2. 31

3. 11

4. 63/2

5. 7c - 8d

6. 49 [By Theorem 141b, $\frac{-6 - 14}{24 - 20} = \frac{14 - a_{13}}{20 - 13}$.]

*

Answers for Part D.

$$1. a_q = a_1 + \sum_{p=1}^{q-1} d = a_1 + d(q-1);$$

$$\sum_{p=1}^q a_p = \sum_{p=1}^q [a_1 + d(p-1)]$$

$$= \sum_{p=1}^q a_1 + \sum_{p=1}^q d(p-1)$$

$$= a_1 q + d \frac{q(q-1)}{2} = \frac{q}{2} [2a_1 + (q-1)d]$$

(b) a' is a sequence of positive integers [Since the domain of a' is I^+ that of $a \circ a'$ will be I^+ if and only if the range of a' is a subset of the domain of a . So, the given condition is a necessary and sufficient one for $a \circ a'$ to be [even] a sequence. If it is satisfied then, for each n , $[a \circ a']_n = a_1 + (a'_n - 1)d = a_1 + [a'_1 + (n - 1)d' - 1]d = a_1 + (a'_1 - 1)d + (n - 1)d'd$. So, in this case, $a \circ a'$ is the AP with 1st term $a_1 + (a'_1 - 1)d$ and common difference $d'd$.]

10. A positive integer n is the sum of m consecutive positive integers if and only if there is a positive integer q such that n is the sum of the first m terms of the AP whose 1st term is q and whose common difference is 1--that is, if and only if $n = \frac{m}{2}[2q + (m - 1) \cdot 1]$, for some $q \in I^+$. Transforming this equation one sees that this is the case if and only if $\frac{2n}{m} - m = 2q - 1$, for some $q \in I^+$. [For related problems, see Exercise ☆5 of Part C on page 8-46 and the accompanying COMMENTARY.]

*

Answers for Part G.

$$1. a_1 = \frac{2s_n - n(n - 1)d}{2n}$$

$$2. d = \frac{2(s_n - na_1)}{n(n - 1)} \quad [n \neq 1]$$

$$3. -2/45$$

$$4. -217/30$$

$$5. 107/15; -1/15$$

$$6. 20$$

7. no [The sum of any number of terms is an even number.]

$$8. a_n = \frac{2s_n - na_1}{n}$$

$$9. a_1 = \frac{2s_n - na_n}{n}$$

[However students solve Exercise 8, they should solve Exercise 9 by noticing that, since (b) is symmetric in ' a_1 ' and ' a_n ', all they need do is interchange ' a_1 ' and ' a_n ' in the solution for Exercise 8.]

Comments on Exercise 10 and 11 of Part G are on TC[8-71, 72]a.

TC[8-69, 70]c

5. The youngest child receives \$430. $[4(2a_1 + 7 \cdot 20) = 4000]$
6. (a) 55 $[(10 \cdot 11)/2]$
- (b) 5; 15 [We want positive integers n (number of rows) and m (number of cases) such that $\frac{n(n+1)}{2} = 24m$. By the quadratic formula, $n = \frac{-1 + \sqrt{1+192m}}{2}$. From a table of squares we find that the first positive integral multiple of 192 which is 1 less than a perfect square is the 5th. For $m=5$, $n=15$.]
7. (a), d ; (b), $7d$; (d) $3d$; (e), 0 ; (f), $-3d$; (g), $d/2$
- [From Unit 5 (see page 5-121), if f and g are linear functions, so is $f \circ g$. By the same reasoning, if f is a linear function and a is an AP then $f \circ a$ is an AP. Proof: Suppose that, for each x , $f(x) = wx + y$, and that, for each n , $a_n = a_1 + (n-1)d$. Then for each n ,
- $$[f \circ a](n) = f(a_n) = w[a_1 + (n-1)d] + y = (wa_1 + y) + (n-1)(wd).$$
- So, $f \circ a$ is an AP with first term $wa_1 + y$ and common difference wd . The reasoning applies even if $w = 0$, in which case f is not a linear function but a constant. This takes care of parts (a), (b), (d), (e), (f), and (g). For part (c), $[h \circ a](p) = [a_1 + (p-1)d]^2$ and, for part (h), $[v \circ a](p) = 1/[a_1 + (p-1)d]$. It is easy algebra to prove that neither $\Delta[h \circ a]$ nor $\Delta[v \circ a]$ is a constant. So, neither $h \circ a$ nor $v \circ a$ is an AP.]
8. 4, 9, 14 [Students should be prepared to ignore the conventional notation for APs when convenient. For example, in this exercise one should look for numbers x and y such that $(x-y) + x + (x+y) = 27$ and $(x-y)x(x+y) = 504$.]
9. (a) yes; $d + d'$ [Since, for each n , $a_n = a_1 + (n-1)d$ and $a'_n = a'_1 + (n-1)d'$, it follows that, for each n , $(a + a')_n = (a_1 + a'_1) + (n-1)(d + d')$. So, $a + a'$ is the AP with 1st term $a_1 + a'_1$ and common difference $d + d'$.

Answers for Part F.

1. [The problem is this:

Al Moore is trying to decide between two job offers. Each position starts at the rate of \$4000 per year. The first offer promises a salary increase of \$100 each six months. The second assures him of an annual increase of \$400. If he plans on staying with one position for five years, which one will pay him more money?]

the first job [For the first job, the number of dollars earned in 5 years is the sum of the first 10 terms of the AP with 1st term 2000 and common difference 100. For the second job, the number of dollars earned in 5 years is the sum of the first 5 terms of the AP with 1st term 4000 and common difference 400. And $5[2 \cdot 2000 + 9 \cdot 100] > \frac{5}{2}[2 \cdot 4000 + 4 \cdot 400].$]

2. 6633 $[= \frac{66}{2}(3 + 198)]$

3. 10732 [By Exercise 2, the sum of those divisible by 3 is 6633 and, similarly, the sum of those divisible by 5 is 3900. Those which are divisible by both 3 and 5 are just those which are divisible by 15 and, by the same method, the sum of these is 1365. So, the sum of the positive integers less than 200 which are divisible by either 3 or 5 is $6633 + 3900 - 1365$ --that is, 9168. Since the sum of all the positive integers less than 200 is $(199 \cdot 200)/2$, the sum of those which are divisible by neither 3 nor 5 is $19900 - 9168.$]

4. \$30.50 [The \$2 rate will be attained at the nth hundred, where $5 + (n - 1) \cdot -\frac{1}{2} = 2$ --that is, at the 7th hundred. So, the dollar-cost of 1000 programs is $\frac{7}{2}(5 + 2) + 2 \cdot 3.$]

1911. 1912. 1913. 1914. 1915. 1916. 1917. 1918. 1919. 1920. 1921. 1922. 1923. 1924. 1925. 1926. 1927. 1928. 1929. 1930. 1931. 1932. 1933. 1934. 1935. 1936. 1937. 1938. 1939. 1940. 1941. 1942. 1943. 1944. 1945. 1946. 1947. 1948. 1949. 1950. 1951. 1952. 1953. 1954. 1955. 1956. 1957. 1958. 1959. 1960. 1961. 1962. 1963. 1964. 1965. 1966. 1967. 1968. 1969. 1970. 1971. 1972. 1973. 1974. 1975. 1976. 1977. 1978. 1979. 1980. 1981. 1982. 1983. 1984. 1985. 1986. 1987. 1988. 1989. 1990. 1991. 1992. 1993. 1994. 1995. 1996. 1997. 1998. 1999. 2000. 2001. 2002. 2003. 2004. 2005. 2006. 2007. 2008. 2009. 2010. 2011. 2012. 2013. 2014. 2015. 2016. 2017. 2018. 2019. 2020. 2021. 2022. 2023. 2024. 2025. 2026. 2027. 2028. 2029. 2030. 2031. 2032. 2033. 2034. 2035. 2036. 2037. 2038. 2039. 2040. 2041. 2042. 2043. 2044. 2045. 2046. 2047. 2048. 2049. 2050. 2051. 2052. 2053. 2054. 2055. 2056. 2057. 2058. 2059. 2060. 2061. 2062. 2063. 2064. 2065. 2066. 2067. 2068. 2069. 2070. 2071. 2072. 2073. 2074. 2075. 2076. 2077. 2078. 2079. 2080. 2081. 2082. 2083. 2084. 2085. 2086. 2087. 2088. 2089. 2090. 2091. 2092. 2093. 2094. 2095. 2096. 2097. 2098. 2099. 2100. 2101. 2102. 2103. 2104. 2105. 2106. 2107. 2108. 2109. 2110. 2111. 2112. 2113. 2114. 2115. 2116. 2117. 2118. 2119. 2120. 2121. 2122. 2123. 2124. 2125. 2126. 2127. 2128. 2129. 2130. 2131. 2132. 2133. 2134. 2135. 2136. 2137. 2138. 2139. 2140. 2141. 2142. 2143. 2144. 2145. 2146. 2147. 2148. 2149. 2150. 2151. 2152. 2153. 2154. 2155. 2156. 2157. 2158. 2159. 2160. 2161. 2162. 2163. 2164. 2165. 2166. 2167. 2168. 2169. 2170. 2171. 2172. 2173. 2174. 2175. 2176. 2177. 2178. 2179. 2180. 2181. 2182. 2183. 2184. 2185. 2186. 2187. 2188. 2189. 2190. 2191. 2192. 2193. 2194. 2195. 2196. 2197. 2198. 2199. 2200. 2201. 2202. 2203. 2204. 2205. 2206. 2207. 2208. 2209. 2210. 2211. 2212. 2213. 2214. 2215. 2216. 2217. 2218. 2219. 2220. 2221. 2222. 2223. 2224. 2225. 2226. 2227. 2228. 2229. 2230. 2231. 2232. 2233. 2234. 2235. 2236. 2237. 2238. 2239. 2240. 2241. 2242. 2243. 2244. 2245. 2246. 2247. 2248. 2249. 2250. 2251. 2252. 2253. 2254. 2255. 2256. 2257. 2258. 2259. 2260. 2261. 2262. 2263. 2264. 2265. 2266. 2267. 2268. 2269. 2270. 2271. 2272. 2273. 2274. 2275. 2276. 2277. 2278. 2279. 2280. 2281. 2282. 2283. 2284. 2285. 2286. 2287. 2288. 2289. 2290. 2291. 2292. 2293. 2294. 2295. 2296. 2297. 2298. 2299. 2300. 2301. 2302. 2303. 2304. 2305. 2306. 2307. 2308. 2309. 2310. 2311. 2312. 2313. 2314. 2315. 2316. 2317. 2318. 2319. 2320. 2321. 2322. 2323. 2324. 2325. 2326. 2327. 2328. 2329. 2330. 2331. 2332. 2333. 2334. 2335. 2336. 2337. 2338. 2339. 2340. 2341. 2342. 2343. 2344. 2345. 2346. 2347. 2348. 2349. 2350. 2351. 2352. 2353. 2354. 2355. 2356. 2357. 2358. 2359. 2360. 2361. 2362. 2363. 2364. 2365. 2366. 2367. 2368. 2369. 2370. 2371. 2372. 2373. 2374. 2375. 2376. 2377. 2378. 2379. 2380. 2381. 2382. 2383. 2384. 2385. 2386. 2387. 2388. 2389. 2390. 2391. 2392. 2393. 2394. 2395. 2396. 2397. 2398. 2399. 2400. 2401. 2402. 2403. 2404. 2405. 2406. 2407. 2408. 2409. 2410. 2411. 2412. 2413. 2414. 2415. 2416. 2417. 2418. 2419. 2420. 2421. 2422. 2423. 2424. 2425. 2426. 2427. 2428. 2429. 2430. 2431. 2432. 2433. 2434. 2435. 2436. 2437. 2438. 2439. 2440. 2441. 2442. 2443. 2444. 2445. 2446. 2447. 2448. 2449. 2450. 2451. 2452. 2453. 2454. 2455. 2456. 2457. 2458. 2459. 2460. 2461. 2462. 2463. 2464. 2465. 2466. 2467. 2468. 2469. 2470. 2471. 2472. 2473. 2474. 2475. 2476. 2477. 2478. 2479. 2480. 2481. 2482. 2483. 2484. 2485. 2486. 2487. 2488. 2489. 2490. 2491. 2492. 2493. 2494. 2495. 2496. 2497. 2498. 2499. 2500. 2501. 2502. 2503. 2504. 2505. 2506. 2507. 2508. 2509. 2510. 2511. 2512. 2513. 2514. 2515. 2516. 2517. 2518. 2519. 2520. 2521. 2522. 2523. 2524. 2525. 2526. 2527. 2528. 2529. 2530. 2531. 2532. 2533. 2534. 2535. 2536. 2537. 2538. 2539. 2540. 2541. 2542. 2543. 2544. 2545. 2546. 2547. 2548. 2549. 2550. 2551. 2552. 2553. 2554. 2555. 2556. 2557. 2558. 2559. 2560. 2561. 2562. 2563. 2564. 2565. 2566. 2567. 2568. 2569. 2570. 2571. 2572. 2573. 2574. 2575. 2576. 2577. 2578. 2579. 2580. 2581. 2582. 2583. 2584. 2585. 2586. 2587. 2588. 2589. 2590. 2591. 2592. 25

1. *Phragmites australis* (Cav.) Trin. ex Steud.

For $d = 0$, for each p , $a_p = a_1$. So, for $d = 0$,

$$\sum_{p=1}^q \frac{1}{\sqrt{a_p} + \sqrt{a_{p+1}}} = \frac{q}{\sqrt{a_1} + \sqrt{a_1}} = \frac{q}{\sqrt{a_1} + \sqrt{a_{q+1}}}.$$


[Proof of algebra theorem: For $a \geq 0$ and $0 \leq b \neq a$, it follows that $\sqrt{b} \neq \sqrt{a}$ --so, $\sqrt{a} - \sqrt{b} \neq 0$. Since, also, $\sqrt{b} \geq 0$ and $\sqrt{a} \geq 0$, it follows that $\sqrt{a} + \sqrt{b} \neq 0$. Consequently,

$$\frac{1}{\sqrt{a} + \sqrt{b}} = \frac{1}{\sqrt{a} + \sqrt{b}} \cdot \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}} = \frac{\sqrt{a} - \sqrt{b}}{(\sqrt{a})^2 - (\sqrt{b})^2} = \frac{\sqrt{a} - \sqrt{b}}{a - b}.]$$

*

Answers for Part ☆J.

$$1. \quad \sum_{p=1}^q [a_1 + (p-1)d] = \sum_{p=1}^q [a_1 + (q+1-p-1)d] = \sum_{p=1}^q [a_1 + (q-p)d]$$


 Theorem 142

The rest of Part ☆J is treated on TC[8-73, 74]a.

$m \neq n$ then $2a_1 + (m+n-1)d = 0$. [$m(m-1) - n(n-1) = (m^2 - n^2) - (m-n) = (m-n)(m+n-1)$]. Consequently, $\frac{m+n}{2}[2a_1 + (m+n-1)d] = 0$ --that

$$\text{is, } \sum_{p=1}^{m+n} a_p = 0.$$

*

Answers for Part ★I.

[The requirement that the terms of a be positive serves the purpose of ensuring, for Exercise 1, that they be nonzero and, for Exercise 2, that they be nonnegative and not all 0.]

1. For each p , $\frac{1}{a_p} - \frac{1}{a_{p+1}} = \frac{a_{p+1} - a_p}{a_p a_{p+1}} = \frac{d}{a_p a_{p+1}}$. So, for $d \neq 0$,

$$\begin{aligned} \sum_{p=1}^q \frac{1}{a_p a_{p+1}} &= \frac{1}{d} \sum_{p=1}^q \left(\frac{1}{a_p} - \frac{1}{a_{p+1}} \right) = \frac{1}{d} \left(\frac{1}{a_1} - \frac{1}{a_{q+1}} \right) \\ &= \frac{1}{d} \left(\frac{a_{q+1} - a_1}{a_1 a_{q+1}} \right) = \frac{q}{a_1 a_{q+1}}. \end{aligned}$$

For $d = 0$, for each p , $a_p = a_1$. So, for $d = 0$,

$$\sum_{p=1}^q \frac{1}{a_p a_{p+1}} = \frac{q}{a_1 a_1} = \frac{q}{a_1 a_{q+1}}.$$

2. For $d \neq 0$, for each p , $a_{p+1} \neq a_p$. So [by the algebra theorem in the hint],

$$\begin{aligned} \sum_{p=1}^q \frac{1}{\sqrt{a_p} + \sqrt{a_{p+1}}} &= \frac{1}{d} \sum_{p=1}^q (\sqrt{a_p} - \sqrt{a_{p+1}}) = \frac{1}{d} (\sqrt{a_1} - \sqrt{a_{q+1}}) \\ &= \frac{1}{d} \frac{a_{q+1} - a_1}{\sqrt{a_{q+1}} + \sqrt{a_1}} = \frac{q}{\sqrt{a_1} + \sqrt{a_{q+1}}}. \end{aligned}$$

$$10. \quad 0 \quad [a_{10} = \frac{2 \cdot 5}{10} - 1 = 0]$$

$$11. \quad 24/11; \text{insufficient information} \quad [(a_1 + a_n)/2 = s_n/n]$$

*

Answers for Part H.

$$1. \quad (a) -848 \quad (b) -56\sqrt{2} \quad (c) 488/3 \quad (d) -3344 \quad (e) 0$$

[Part (e) of Exercise 1 may prepare students for Exercise ☆2. The condition stated in part (e) amounts to the assertion that $5(2a_1 + 9d) = 3(2a_1 + 5d)$ --that is, that $2a_1 + 15d = 0$. Since the sum of the first 16 terms of the AP is $8(2a_1 + 15d)$, this sum is 0. To see what is going on, select a value for ' a_1 '--say, to simplify the arithmetic, 15--and compute the corresponding value of ' d '--in this case, -2. The first few terms of this AP are

$$15, 13, 11, 9, 7, 5, 3, 1, -1, -3, -5.$$

Clearly, the sum of the 7th through the 10th term is 0--as is to be expected, since the AP was chosen so that the sum of its first 6 terms equals the sum of its first 10 terms. Furthermore, the sum of the 6th through the 11th term is 0, as is the sum of the 5th through the 12th, that of the 4th through the 13th, ..., and that of the 1st through the 16th. The possibility of such a situation occurring is a consequence of the fact that the sum-sequence of an AP is a subset of a quadratic function-- $\forall_n s_n = \frac{d}{2}n^2 + \left(a_1 - \frac{d}{2}\right)n$. The situation will occur whenever the axis of symmetry of this function is, for some $q \in \mathbb{I}^+$, $\{(x, y): x = \frac{2q+1}{2}\}$. Exercise ☆13 in page 8-3 deals with a related situation.]

$$\begin{aligned} \star 2. \quad & \text{If } \sum_{p=1}^m a_p = \sum_{p=1}^n a_p \text{ then } \frac{m}{2}[2a_1 + (m-1)d] = \frac{n}{2}[2a_1 + (n-1)d] \text{--that is,} \\ & (m-n)(2a_1) + [m(m-1) - n(n-1)]d = 0. \text{ It follows that if, in addition,} \end{aligned}$$

*

If a is the AP such that $a_1 = 6$ and $a_7 = 15$, it follows by Theorem 141b that $d = \frac{15 - 6}{7 - 1} = \frac{3}{2}$. So, the 5 arithmetic means between 6 and 15 are 7.5, 9, 10.5, 12, and 13.5. [Instead of using Theorem 141b, one can calculate the difference, d , between successive ones of n arithmetic means between x and y by the formula ' $d = \frac{y - x}{n + 1}$ '.]

*

Answers for Part K.

1. 6, 8 2. 2.5, 4, 5.5 3. 7.6, 6.8, 6, 5.2, 4.4
4. -6.8, -7.6, -8.4, -9.2, -10, -10.8, -11.6, -12.4, -13.2
5. 0, -1.5 6. 10 7. $(x + y)/2$
8. $\frac{3x + y}{4}$, $\frac{x + y}{2}$, $\frac{x + 3y}{4}$
9. $\frac{(n + 1 - m)x + my}{n + 1}$, for $1 \leq m \leq n$.

*

The phrase 'arithmetic means' [note plural] is ordinarily used as in Part K. The phrase 'the arithmetic mean' may be used either in a context 'the arithmetic mean of the sequence...', as in the text or, derivatively, in a context 'the arithmetic mean of the numbers...' [see last paragraph of page 8-74]. In both cases, as in 'arithmetic progression', the proper pronunciation is ar'ith-met'ic, not a-rith'me-tic.

$$\begin{aligned}
 2. \quad 2 \sum_{p=1}^q [a_1 + (p-1)d] &= \sum_{p=1}^q [a_1 + (p-1)d] + \sum_{p=1}^q [a_1 + (q-p)d] \quad [\text{Exercise 1}] \\
 &= \sum_{p=1}^q [a_1 + (p-1)d + a_1 + (q-p)d] \\
 &= \sum_{p=1}^q [2a_1 + (q-1)d] = q[2a_1 + (q-1)d]
 \end{aligned}$$

$$\text{So, } \forall_n \sum_{p=1}^n [a_1 + (p-1)d] = \frac{n}{2} [2a_1 + (n-1)d].$$

$$3. \quad (i) \quad \sum_{i=j}^{j-1} a_i = 0 = \sum_{i=j}^{j-1} a_{j-1+j-i}$$

$$(ii) \quad \text{Suppose [for some } k \geq j-1 \text{] that } \sum_{i=j}^k a_i = \sum_{i=j}^k a_{k+j-i}. \quad \text{Then,}$$

$$\sum_{i=j}^{k+1} a_i = \sum_{i=j}^k a_i + a_{k+1} = \sum_{i=j}^k a_{k+j-i} + a_{k+1}$$

$$= a_{k+1} + \sum_{i=j}^k a_{k+j-i} = \sum_{i=j-1}^k a_{k+j-i}$$

$$= \sum_{i=j}^{k+1} a_{k+j-i-1} = \sum_{i=j}^{k+1} a_{k+1+j-i}.$$

Etc.

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DEPARTMENT OF THE HISTORY OF ARTS
AND ARCHITECTURE

CHICAGO, ILLINOIS

TO THE HONORABLE THE PRESIDENT OF THE UNIVERSITY OF CHICAGO
AND THE FACULTY OF THE HISTORY OF ARTS AND ARCHITECTURE

THE FOLLOWING IS A LIST OF THE

MEMBERS

OF THE

COMMISSION ON THE HISTORY OF ARTS AND ARCHITECTURE

AND OF THE

COMMISSION ON THE HISTORY OF LITERATURE

AND OF THE

COMMISSION ON THE HISTORY OF SCIENCE

AND OF THE

COMMISSION ON THE HISTORY OF PHILOSOPHY

(c) If a is a constant sequence and b is any sequence then

$$(\overline{ab})_q = \frac{1}{q} \sum_{p=1}^q (ab)_p = \frac{1}{q} \sum_{p=1}^q a_p b_p = \frac{1}{q} \sum_{p=1}^q a_1 b_p = a_1 \cdot \frac{1}{q} \sum_{p=1}^q b_p \\ = a_1 (\overline{b})_q = a_q (\overline{b})_q. \text{ So, } \overline{ab} = a\overline{b}.$$

3. 92.8125 $\left[= \frac{100 \cdot 3 + 95 \cdot 4 + 90 \cdot 8 + 85 \cdot 1}{3 + 4 + 8 + 1} \right]$ [If you wish to, this is an appropriate time to make a few remarks about frequency and related statistical concepts. See Part ★M which follows.]

*

Answers for Part ★M.

1. (a) 10.5 (b) 0

$$2. \forall_x \forall_n \left[\sum_{p=1}^n (a_p - x) = 0 \iff x = (\overline{a})_n \right]; \sum_{p=1}^q (a_p - c) = \sum_{p=1}^q a_p - cq = 0 \\ \text{if and only if } c = \frac{1}{q} \sum_{p=1}^q a_p.$$

$$3. f(x) = \sum_{p=1}^n (a_p^2 - 2a_p x + x^2) = \sum_{p=1}^n a_p^2 - 2x \sum_{p=1}^n a_p + nx^2, \text{ for each } x.$$

The argument for which this quadratic function has its minimum

$$\text{value is } \left(2 \sum_{p=1}^n a_p \right) / (2n) \text{--that is, } (\overline{a})_n.$$

Mathematics

$$\frac{1}{x^2} = x^{-2} \Rightarrow \frac{d}{dx} x^{-2} = -2x^{-3} = -\frac{2}{x^3}$$

$$\frac{d}{dx} \left(\frac{1}{x^2} \right) = -\frac{2}{x^3}$$

$$\frac{d}{dx} x^{-2} = -2x^{-3}$$

$$\frac{d}{dx} \left(\frac{1}{x^2} \right) = -\frac{2}{x^3}$$

Mathematics

$$\frac{d}{dx} \left(\frac{1}{x^2} \right) = -\frac{2}{x^3}$$

$$\frac{d}{dx} x^{-2} = -2x^{-3}$$

$$\frac{d}{dx} \left(\frac{1}{x^2} \right) = -\frac{2}{x^3}$$

Answers for Part L.

$$1. (a) \quad 13/2 \left[\forall_n \frac{1}{n} \sum_{p=1}^n p = \frac{1}{n} \cdot \frac{n(n+1)}{2} \right]$$

$$(b) \quad n \quad \cdot \left[\forall_n \frac{1}{n} \sum_{p=1}^n (2p - 1) = \frac{1}{n} \cdot n^2 \right]$$

$$(c) \quad n+1 \left[\forall_n \frac{1}{n} \sum_{p=1}^n 2p = \frac{1}{n} \cdot n(n+1) \right]$$

$$(d) \quad (a)_q = \frac{1}{q} \sum_{p=1}^q a_p = \frac{1}{q} \cdot \frac{q}{2} (a_1 + a_q) = \frac{a_1 + a_q}{2}$$

2. (a) If a is a constant sequence then, for each p , $a_p = a_1$. Hence,

$$(\bar{a})_q = \frac{1}{q} \sum_{p=1}^q a_p = \frac{1}{q} \sum_{p=1}^q a_1 = \frac{1}{q} (qa_1) = a_1 = a_q. \quad \text{So, } \forall_n (\bar{a})_n = a_n$$

--that is, $\bar{a} = a$.

$$(b) \quad \text{For any sequences } a \text{ and } b, (\overline{a+b})_q = \frac{1}{q} \sum_{p=1}^q (a+b)_p = \frac{1}{q} \sum_{p=1}^q (a_p + b_p)$$

$$= \frac{1}{q} \sum_{p=1}^q a_p + \frac{1}{q} \sum_{p=1}^q b_p = (\bar{a})_q + (\bar{b})_q. \quad \text{So, } \overline{a+b} = \bar{a} + \bar{b}.$$

3. Since $1 + 4p > 9 + 2p$ if and only if $p > 4$, it follows from Theorem

143 that, for each $n \geq 5$, $\sum_{p=5}^n (1 + 4p) > \sum_{p=5}^n (9 + 2p)$. Since, for

$m \leq 4$, $\sum_{p=m}^m (1 + 4p) = 1 + 4m \not> 9 + 2m = \sum_{p=m}^m (9 + 2p)$, 5 is the

smallest value of 'm' which can be used.

the first of these is the fact that the
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the thirty-ninth is the fact that the
the fortieth is the fact that the

Part (i) of an inductive proof of Theorem 143:

Suppose that $\forall_{m \leq 1} a_m < b_m$. Since $1 \in I^+$ and $1 \leq 1$, it follows that

$a_1 < b_1$. Since, by the recursive definition for Σ -notation, $\sum_{p=1}^1 a_p = a_1$

and $\sum_{p=1}^1 b_p = b_1$, it follows that $\sum_{p=1}^1 a_p < \sum_{p=1}^1 b_p$.

Hence, if $\forall_{m \leq 1} a_m < b_m$ then $\sum_{p=1}^1 a_p < \sum_{p=1}^1 b_p$.

*

The justification of the statement made on lines 6 and 5 from the bottom of page 8-77 is that, by the if-part of Theorem 107b, if $m \leq q$ then $m < q + 1$ [and, if $m = q + 1$ then $m \leq q + 1$].

*

Answers for Part A.

1. Since, for each m , $2m + 1 > 2m$ [Theorem 82 and atpi], it follows that, for each m , $\frac{1}{2m+1} < \frac{1}{2m}$ [Theorem 100, Theorem 82, etc.].

Hence, by Theorem 143, $\forall_n \sum_{p=1}^n \frac{1}{2p+1} < \sum_{p=1}^n \frac{1}{2p}$.

2. [Similar to Exercise 1. Since, for each m , $\frac{4m-3}{2m-1} < \frac{4m-1}{2m}$, the theorem follows by Theorem 143. [$\frac{4m-3}{2m-1} < \frac{4m-1}{2m}$ because $2m > 0$, $2m-1 > 0$, and $2m(4m-3) < (2m-1)(4m-1)$.]]

Answers for Part C.

1. 3

2. [See page 8-81.]

3. 50 [There are ten categories to consider: boxes containing 231 apples, those containing 232 apples, . . . , and those containing 240 apples. If each category contained at most 49 boxes then there would be at most 490 boxes in all. So, since there are more than 490 boxes, some category contains at least 50 boxes.]

*

When one answers 'eleven' for Exercise 1 of Part C, he may be thinking of how many socks might have to be taken out to find two that don't match.

The analyses of Exercises 1 and 2 may seem pretty heavy for such simple problems. They do point out the similarity between the two problems, and suggest a method for solving more difficult problems of this kind.

5. (a) 1998 [By the theorem of Exercise 4,

$$2\sqrt{10^6} - 2 < \sum_{p=1}^{10^6} \frac{1}{\sqrt{p}} < 2\sqrt{10^6} - 1$$

$$\text{--that is, } 1998 < \sum_{p=1}^{10^6} \frac{1}{\sqrt{p}} < 1999.]$$

(b) $2n - 2$ [As for part (a), with ' n^2 ' in place of ' 10^6 '.]

☆6. [The same procedure used in proving the theorem of Exercise 4 leads to a more general theorem:

$$\forall_m \forall_{n > m} 2\sqrt{n} - 2\sqrt{m} < \sum_{p=m}^n \frac{1}{\sqrt{p}} < 2\sqrt{n} - 2\sqrt{m} + \frac{1}{\sqrt{m}}$$

In particular,

$$\forall_m \forall_{n > m} 2(n - m) < \sum_{p=m}^{n^2} \frac{1}{2\sqrt{p}} < 2(n - m) + \frac{1}{m}.]$$

(a) 90000 [By the result just stated,

$$2 \cdot 900 < \sum_{p=10^4}^{10^6} \frac{1}{\sqrt{p}} < 2 \cdot 900 + \frac{1}{100}.$$

$$\text{So, } 90000 < 50 \sum_{p=10^4}^{10^6} \frac{1}{\sqrt{p}} < 90000 + \frac{1}{2}.]$$

(b) $2q(n - m)$ [Similar to part (a).]

✱

It now follows, by a similar extension of Theorem 138, that

$$\sqrt{q+1} - \sqrt{k} < \sum_{p=k}^q \frac{1}{2\sqrt{p}} < \sqrt{q} - \sqrt{k-1}.$$

Consequently,

$$\forall_m \forall_{n \geq m} \sqrt{n+1} - \sqrt{m} < \sum_{p=m}^n \frac{1}{2\sqrt{p}} < \sqrt{n} - \sqrt{m-1}.$$

4. [The answer is given largely in the hint.] By the left inequation in the theorem of Exercise 3, since $n \geq 1$,

$$\sqrt{n+1} - \sqrt{1} < \sum_{p=1}^n \frac{1}{2\sqrt{p}}.$$

Since, by Theorems 90 and 98b, $\sqrt{n} < \sqrt{n+1}$, it follows that $\sqrt{n} - 1 < \sqrt{n+1} - 1$. Hence, using Theorem 133 and the mtpi,

$$2\sqrt{n} - 2 < \sum_{p=1}^n \frac{1}{\sqrt{p}}.$$

Now, by the right inequation in the theorem of Exercise 3, for $n > 1$,

$$\sum_{p=2}^n \frac{1}{2\sqrt{p}} < \sqrt{n} - \sqrt{1}.$$

Hence, by Theorem 136, $\sum_{p=1}^n \frac{1}{2\sqrt{p}} < \sqrt{n} - 1 + \frac{1}{2}$.

So, $\sum_{p=1}^n \frac{1}{\sqrt{p}} < 2\sqrt{n} - 1.$

Combining these two results [recalling that the second holds only for $n > 1$], we obtain the desired theorem.

As motivation for Part B on page 8-79, reconsider Problem IV on page 7-1 of Unit 7:

Study these sentences and complete the last one.

$$\begin{array}{ll} 2 < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} < 3 & 4 < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{9}} < 5 \\ 6 < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{16}} < 7 & 8 < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{25}} < 9 \\ 18 < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{100}} < 19 & < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{106}} < \end{array}$$

*

Answers for Part B.

1. By the theorem referred to in the hint [which is proved in the COM-MENTARY for pages 8-71 and 8-72] it follows that, for $c - 1 \geq 0$,

$$\frac{1}{\sqrt{c} + \sqrt{c-1}} = \frac{\sqrt{c} - \sqrt{c-1}}{c - (c-1)} = \sqrt{c} - \sqrt{c-1}.$$

Also, since $c - 1 < c$, it follows, for $c - 1 \geq 0$, that $\sqrt{c-1} < \sqrt{c}$. Hence, $\sqrt{c} + \sqrt{c-1} < 2\sqrt{c}$, and $\frac{1}{2\sqrt{c}} < \frac{1}{\sqrt{c} + \sqrt{c-1}}$. So,

$$\forall_{x \geq 1} \frac{1}{2\sqrt{x}} < \sqrt{x} - \sqrt{x-1}.$$

2. [Similar to Exercise 1. For $c \geq 0$, $1/(2\sqrt{c}) > 1/(\sqrt{c+1} + \sqrt{c}) = \sqrt{c+1} - \sqrt{c}$.]

3. Since $p \geq 1$, it follows from the theorems of Exercises 1 and 2 that

$$\sqrt{p+1} - \sqrt{p} < \frac{1}{2\sqrt{p}} < \sqrt{p} - \sqrt{p-1}.$$

So, by a slight extension of Theorem 143, for $q \geq k$,

$$\sum_{p=k}^q (\sqrt{p+1} - \sqrt{p}) < \sum_{p=k}^q \frac{1}{2\sqrt{p}} < \sum_{p=k}^q (\sqrt{p} - \sqrt{p-1}).$$

$$\forall_x \forall_n \left[\sum_{p=1}^n a_p > xn \Rightarrow \exists_{m \leq n} a_m > x \right].$$

*

The proof of Theorem 144 is like that just given for the "corollary" except that Theorem 143 is used as given on page 8-76['<' instead of, as above, ' \leq '], and in place of 'c' one should write 'c/q'.

The explanation asked for concerning line 4 on page 8-81 is that, since

the numbers a_p are integers, so is $\sum_{p=1}^2 a_p$. And [for the antecedent],

an integer is greater than 2 if and only if it is greater than or equal to 3, and [for the conclusion] is greater than 1 if and only if it is greater than or equal to 2. [For positive integers these last two assertions are consequences of Theorem 106. For integers in general they follow from a similar theorem ('j' and 'k' in place of 'm' and 'n')].

*

The corollary of Theorem 143 which is displayed as the 4th line from the bottom of page 8-81, is proved as follows:

By [one of the forms of] Theorem 143 [see remark preceding 'EXERCISES' on page 8-78], it follows that

$$\forall m \leq q \ a_m \leq c \Rightarrow \sum_{p=1}^q a_p \leq \sum_{p=1}^q c.$$

Since $\sum_{p=1}^q c = cq$, it follows [by contraposition] that

$$\sum_{p=1}^q a_p \not\leq cq \Rightarrow \exists m \leq q \ a_m \not\leq c.$$

So, by Theorem 88,

$$\sum_{p=1}^q a_p > cq \Rightarrow \exists m \leq q \ a_m > c. \quad \text{Consequently,}$$

The sentence (*) on page 8-81 follows from the sentence at the foot of page 8-80 in this way. First, by contraposition, one derives from the latter sentence:

$$\text{not} - \left(\sum_{p=1}^2 a_p \leq 2 \right) \Rightarrow \text{not} - \left(\forall_{m \leq 2} a_m \leq 1 \right)$$

Now the antecedent can, as usual, be written ' $\sum_{p=1}^2 a_p \not\leq 2$ '. And, from

the meaning of 'for each...' [' $\forall_{...}$ '] and 'there is some...such that' [' $\exists_{...}$ '] it follows that the conclusion is equivalent to ' $\exists_{m \leq 2} a_m \not\leq 1$ '. By virtue of Theorem 88, we can replace each ' $\not\leq$ ' by '>'. Doing so, we obtain (*). In dealing with the equivalence of 'not for each...' and 'there is some...such that not' you may find it useful to give [and ask for] nonmathematical illustrations. For examples:

- { It is not the case that each student in this room is less than _____ years old.
- { There is some student in this room who is not less than _____ years old.
- { Not every cow is brown.
- { Some cow is not brown.

Using the rules of reasoning adopted in the Appendix of Unit 6 and two additional rules for handling existential generalizations, we can prove that each sentence of the form:

$$\text{not} - \forall_x F(x) \iff \exists_x \text{not} - F(x)$$

is logically valid.

*

- ★(d) You might never find two such terms. [The AP whose 1st term is 3 and whose common difference is 4 has no term whose remainder on dividing by 3 equals its remainder on dividing by 4. For, the remainder upon dividing any term by 4 is 3, and 3 is never a remainder upon dividing by 3.]

*

Answers for ★E.

1. There are just n remainders possible upon dividing a positive integer by n . So, if the members of S are the numbers a_p , for $p \leq n$,

then either, for some q , the remainder upon dividing $\sum_{p=1}^q a_p$ by n is

0, or for some q_1 and $q_2 > q_1$, the remainders upon dividing $\sum_{p=1}^{q_1} a_p$

and $\sum_{p=1}^{q_2} a_p$ are the same. In the latter case, the remainder upon

dividing $\sum_{p=1}^{q_2} a_p - \sum_{p=1}^{q_1} a_p$ --that is, $\sum_{p=q_1+1}^{q_2} a_p$ --by n is 0. So, in

any case, some sum of members of S is divisible by n .

2. If two positive integers have the same greatest odd factor then the larger of the two is divisible by the smaller--in fact, the quotient is some power of 2. Now the greatest odd factor of a positive integer less than or equal to $2n$ is one of the n odd numbers which are less than or equal to $2n$. So, given more than n --say, $n+1$ --positive integers less than or equal to $2n$, at least two of these have the same greatest odd factor. And, as noted above, of these two, one is a factor of the other.

4. (a) 4th term; 5th; 6th; 7th [There are q possible remainders upon dividing positive integers by a positive integer q . If, for $p \leq q$, a_p is the number of terms among the first m terms of the sequence which have the remainder

$$p - 1 \text{ upon dividing by } q, \text{ then } \sum_{p=1}^q a_p = m \text{ and we wish}$$

to know how large m must be to be sure that, for some $p \leq q$, $a_p \geq 2$. By Theorem 144, this will be the case if $m/q > 1$ --that is if $m \geq q + 1$. So, by the $(q + 1)$ th term, one will find two terms which have the same remainder upon dividing by q .]

- (b) 13th term; 7th; 13th [For two terms to have the same remainder upon dividing by a positive integer q , it is necessary and sufficient that their difference be an integral multiple of q . So, for them to have the same remainder upon division by each of two positive integers, it is necessary and sufficient that their difference be a multiple of each of the two positive integers--that is, be a multiple of the lowest common multiple of the two positive integers. But this is a necessary and sufficient condition that the two terms have the same remainder upon division by the lowest common multiple of the two positive integers.]

- (c) 5th term [By the 4th term one would have found two terms which have the same remainder upon dividing by 3, and by the 5th term one would also have found two terms which have the same remainder upon dividing by 4.]

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Answers for Part D.

1. Exercise 2: For each $p \leq 240$ let a_p be the number of boxes which contain exactly p apples. Then, $\sum_{p=1}^{240} a_p = 500$ and, by Theorem 144, $\exists_{m \leq 240} a_m \geq \frac{500}{240}$. Since the numbers a_p are integers, $\exists_{m \leq 240} a_m \geq 3$. So at least 3 boxes contain the same number of apples.

Exercise 3: In this case, $\sum_{p=231}^{240} a_p = 500$ and, by an obvious extension of Theorem 144, there is an m [$231 \leq m \leq 240$] such that $a_m \geq \frac{500}{10} = 50$. So, at least 50 boxes contain the same number of apples. [Theorem 144 can be used as it stands by letting a_p , for $p \leq 10$, be the number of boxes which contain exactly $230 + p$ apples.]

2. 5 [Let a_1 be the number who have birthdays in January, and a_2 the number who do not. To be sure, by Theorem 144, that either $a_1 \geq 3$ or $a_2 \geq 3$, it is sufficient to know that $(a_1 + a_2)/2 > 2$ [--that is, that $x/2 > 2$]. So, it is sufficient to have 5 people present.]
3. 136 [To be sure that at least one white bottle contains not less than 10 pills, the number x of pills in the brown bottle whose pills are distributed must be such that $x/3 > 9$. So, some brown bottle must contain at least 28 pills. To be sure of this, the total number of y pills must be such that $y/5 > 27$. So, to be sure of the result, there must be at least 136 pills in the five brown bottles.]

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A way of naming the required sum is by using a sign for double-summation:

$$\sum_{p, q=1}^n pq$$

As the table suggests,

$$\sum_{p, q=1}^n pq = \sum_{p=1}^n \left(\sum_{q=1}^n pq \right) = \sum_{p=1}^n \left(p \sum_{q=1}^n q \right) \stackrel{\text{Theorem 133}}{=} \left(\sum_{q=1}^n q \right) \left(\sum_{p=1}^n p \right)$$

The transformation from double-summation to iterated summation is analogous to the calculus procedure of transforming a double integral to an iterated integral.

13. It may help students if you suggest the following scheme for listing the ordered pairs.

| | | | | | | |
|------------|------------|------------|-----|----------------|----------------|------------|
| (1, n) | (2, n) | (3, n) | ... | (n - 2, n) | (n - 1, n) | (n, n) |
| (1, n - 1) | (2, n - 1) | (3, n - 1) | ... | (n - 2, n - 1) | (n - 1, n - 1) | (n, n - 1) |
| (1, n - 2) | (2, n - 2) | (3, n - 2) | ... | (n - 2, n - 2) | (n - 1, n - 2) | (n, n - 2) |
| \vdots | \vdots | \vdots | | \vdots | \vdots | \vdots |
| (1, 3) | (2, 3) | (3, 3) | ... | (n - 2, 3) | (n - 1, 3) | (n, 3) |
| (1, 2) | (2, 2) | (3, 2) | ... | (n - 2, 2) | (n - 1, 2) | (n, 2) |
| (1, 1) | (2, 1) | (3, 1) | ... | (n - 2, 1) | (n - 1, 1) | (n, 1) |

Then, the sums of the products column by column starting with the leftmost column are

$$1 \cdot \sum_{p=1}^n p \quad 2 \cdot \sum_{p=1}^n p \quad 3 \cdot \sum_{p=1}^n p \quad \dots \quad (n-2) \cdot \sum_{p=1}^n p \quad (n-1) \cdot \sum_{p=1}^n p \quad n \cdot \sum_{p=1}^n p$$

So, the sum of the sums is

$$[1 + 2 + 3 + \dots + (n-2) + (n-1) + n] \cdot \sum_{p=1}^n p,$$

or,

$$\left[\sum_{p=1}^n p \right] \cdot \sum_{p=1}^n p.$$

Hence, the sum is $\frac{n(n+1)}{2} \cdot \frac{n(n+1)}{2}$, or $\frac{n^2(n+1)^2}{4}$.

Some students may just as easily see, directly, that the required sum is

$$(1 + 2 + 3 + \dots + n)(1 + 2 + 3 + \dots + n).$$

is to use Theorem 130:

$$\begin{aligned}
 \frac{q(4q^2 - 1)(12q^2 - 7)}{15} + (2q + 1)^4 &= \frac{2q + 1}{15} [q(2q - 1)(12q^2 - 7) + 15(2q + 1)^3] \\
 &= \frac{2q + 1}{15} [24q^4 + 108q^3 + 166q^2 + 97q + 15] \\
 &= \frac{2q + 1}{15} (q + 1)(24q^3 + 84q^2 + 82q + 15) \\
 &= \frac{(q + 1)(2q + 1)(2q + 3)(12q^2 + 24q + 5)}{15} \\
 &= \frac{(q + 1)[4(q + 1)^2 - 1][12(q + 1)^2 - 7]}{15}
 \end{aligned}$$

[To do the factoring expeditiously, one needs to know the expected result.]

A third procedure is as follows:

$$\begin{aligned}
 \sum_{p=1}^q (2p - 1)^4 &= 16 \sum_{p=1}^q p^4 - 32 \sum_{p=1}^q p^3 + 24 \sum_{p=1}^q p^2 - 8 \sum_{p=1}^q p + \sum_{p=1}^q 1 \\
 &= 16 \frac{q(q + 1)(2q + 1)(3q^2 + 3q - 1)}{30} - 32 \frac{q^2(q + 1)^2}{4} \\
 &\quad + 24 \frac{q(q + 1)(2q + 1)}{6} - 8 \frac{q(q + 1)}{2} + q \\
 &= \frac{q(48q^4 - 40q^2 + 7)}{15} = \frac{q(4q^2 - 1)(12q^2 - 7)}{15}
 \end{aligned}$$

11. (a) $2x^3 - 3x^2 + 4x - 5$

(b) $2x^3 - x^2 + x - 1$

12. (a) $92/13$

(b) 4

10. The simplest procedure, and one which would enable one to discover the theorem as well as prove it, is to note that, for any sequence a ,

$$\forall_n \sum_{p=1}^n a_{2p-1} + \sum_{p=1}^n a_{2p} = \sum_{p=1}^{2n} a_p.$$

[This is easily proved by induction.] Using this, one has:

$$\begin{aligned} \sum_{p=1}^q (2p-1)^4 &= \sum_{p=1}^{2q} p^4 - \sum_{p=1}^q (2p)^4 = \sum_{p=1}^{2q} p^4 - 16 \sum_{p=1}^q p^4 \\ &= \frac{2q(2q+1)(4q+1)(12q^2+6q-1)}{30} \\ &\quad - 16 \frac{q(q+1)(2q+1)(3q^2+3q-1)}{30} \\ &= \frac{q(2q+1)}{15} [(4q+1)(12q^2+6q-1) \\ &\quad - 8(q+1)(3q^2+3q-1)] \\ &= \frac{q(2q+1)}{15} [24q^3 - 12q^2 - 14q + 7] \\ &= \frac{q(2q+1)}{15} [12q^2(2q-1) - 7(2q-1)] \\ &= \frac{q(2q+1)}{15} (12q^2-7)(2q-1) \\ &= \frac{q(4q^2-1)(12q^2-7)}{15}. \end{aligned}$$

If one does not have to discover the theorem, the simplest procedure

circular sectors is $\pi s^2/8$, and that for the larger semi-circular sector is $\pi s^2/2$. Now, looking from two points of view at the region bounded by the closed path AXBYCZDA, we see that

$$K_1 + K_2 + K_3 + \frac{\pi s^2}{2} = 3 \cdot \frac{\pi s^2}{8} + \left(\frac{\pi s^2}{8} + K_4 \right).$$

$$\text{So, } K_1 + K_2 + K_3 = K_4.$$

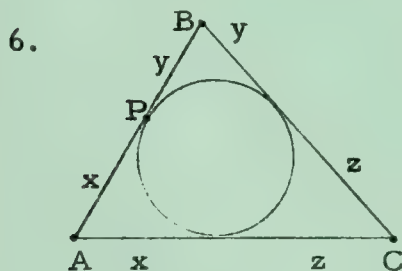
$$2. \text{ almost } 42\frac{1}{2} \text{ yards } \left[\frac{26}{24.5} = \frac{x}{40} \Rightarrow x = 42\frac{22}{49} \right]$$

$$3. (a) -56k^6 + k^5 + 17k^4 + 2k^3 + 40k^2 + 5k$$

$$(b) 21x^6 - 25x^5y + 26x^4y^2 + 6x^3y^3 - 15x^2y^4 - 2xy^5 + y^6$$

4. No. [Substitute from the first equation into the second to get ' $(x - 2)^2 + (2x - 3)^2 = 25$ '. Since the discriminant of this quadratic equation is 496 which is not a perfect square, there are no integral pairs which satisfy both equations.]

$$5. 7fw/c \quad [x/w = 7f/c]$$



$$2x + 2y + 2z = AB + BC + CA = 24;$$

$$x + y + z = 12; \quad y + z = 8; \quad x = 4 = AP$$

$$7. b = \frac{21 - 5a}{4}$$

$$8. 11\frac{1}{9}\%$$

$$9. \text{ impossible } \left[\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2} \text{ and } \frac{1}{a} + \frac{1}{b} = \frac{1}{6} \text{ and } \frac{1}{a} + \frac{1}{c} = \frac{1}{8} \right) \Rightarrow a = -4.8 \right]$$

Answers for Part F.

1. There are at most 23 teachers at Z.H.S. [If there are n teachers and, for $p \leq n$, the p th teacher received a_p letters, while Mr. Jones

received a_{n+1} letters, then $\sum_{p=1}^{n+1} a_p = 49$. By Theorem 144,

$\exists_{m \leq n+1} a_m \geq 49/(n+1)$. For Mr. Jones to conclude that someone had received at least 3 letters, it must be the case that $49/(n+1) > 2$. So, $n+1 \leq 24$.]

2. There are at least 16 teachers at Z.H.S. [For Mr. Jones to be unable to conclude that someone had received more than 3 letters, it must be the case that $49/(n+1) \leq 3$. So, $n+1 \geq 17$.

3. 16 [Supposing that the $(n-1)$ st and n th teachers received no letter,

$$\sum_{p=1}^{n-2} a_p = 49 - 6. \text{ So, by Theorem 144, } \exists_{m \leq n-2} a_m \geq 43/(n-2).$$

Since Mr. Jones could conclude that some teacher had received at least 4 letters, it must be the case that $43/(n-2) > 3$. So, $n \leq 16$. But, by Exercise 2, $n \geq 16$. So, there are just 16 teachers at Z.H.S.

*

Answers for Miscellaneous Exercises.

| | |
|----------------|---|
| <u>Easy:</u> | 2-5, 7, 8, 11, 12, 15-24, 26-32, 33 (tedious), 34, 35, 37, 38, 40-43, 45-52 |
| <u>Medium:</u> | 1, 6, 9, 13, 14, 25, 36, 44 |
| <u>Hard:</u> | 10, 39 |

1. Let s be the side-measure of the regular hexagon. Then, the area-measure of the region bounded by each of the four smaller semi-

29. (a) $2x^3 + 3x^2 - 13x + 6$

(b) $5x^2 - 3x + 1$

30. (a) $2(x^2 + y^2 + z^2 - xy - yz - zx)$

(b) Since $(a - b)^2 \geq 0$, $(b - c)^2 \geq 0$, and $(c - a)^2 \geq 0$, it follows that

$$\begin{aligned} 0 &\leq (a - b)^2 + (b - c)^2 + (c - a)^2 \\ &= 2(a^2 + b^2 + c^2 - ab - bc - ca). \end{aligned}$$

So, since $2 > 0$, $a^2 + b^2 + c^2 - ab - bc - ca \geq 0/2 = 0$.

31. (a) $x = \frac{aqr + brp - cpq}{qr + rp - pq}$

(b) $x = \frac{c + q(a - b)}{p(a - b)}$

(d) $4y^2 + 4y - 3$

(e) $12 + 7c - 12c^2$

(f) $a^2c^2 + 2ac - 3$

21. $b_{14} \quad [5 + 8 \cdot 10 = (p+3)(p-9), \quad p^2 - 6p - 112 = 0, \quad (p-14)(p+8) = 0]$

$$\begin{aligned}
 22. \quad \sum_{p=1}^q p(p+2a-1) &= \sum_{p=1}^q p^2 + (2a-1) \sum_{p=1}^q p \\
 &= \frac{q(q+1)(2q+1)}{6} + (2a-1) \frac{q(q+1)}{2} \\
 &= \frac{q(q+1)}{6} [(2q+1) + 3(2a-1)] \\
 &= \frac{q(q+1)(2q+6a-2)}{6} = \frac{q(q+1)(q+3a-1)}{3}
 \end{aligned}$$

23. $9999^2 - 1^2 = (9999+1)(9999-1) = 99980000$. So, $9999^2 = 99980001$.

24. (a) 12

(b) 13

(c) 4.1

25. Suppose that the first time he tried, he made a solid square with m pennies on each side. Since he had 116 pennies left over, he must have started with $m^2 + 116$ pennies. Since he was 25 short of making a solid square with $m+3$ pennies on each side, it follows that $m^2 + 116 = (m+3)^2 - 25$ --that is, that $m = 22$. So, he must have 600 pennies, and since 600 is not a perfect square, he can't form a solid square with them.

26. (a) $-5xy - 2yz + 5zx$ (b) $-2pq + 10qr - 7rp$ (c) $3a - 4b + 4c - 3d$

27. 2

28. false [The intersection is $\{(1, -1)\}$.]

14. For $a > 3$, it follows that $a - 1 > 2$ and, since $2 > 0$, that $a - 1 > 0$.

So,
$$1 < \frac{a+1}{a-1} < 2$$

$$\iff a - 1 < a + 1 < 2a - 2$$

$$\iff 0 < 2 < a - 1.$$

Since $a - 1 > 2$ and $2 > 0$, it follows that, for $a > 3$, $1 < \frac{a+1}{a-1} < 2$.

15. $11w/(12k)$

16. (a) $4a^6$ (b) a^2b^4 (c) $81p^6q^{12}$ (d) $\frac{1}{9x^6}$ (e) x^9y^6
 (f) $8x^6z^9$ (g) $-27p^{12}q^6$ (h) $= \frac{216}{125a^{15}b^6}$

17. (a) a (b) a^2 (c) $\frac{a^4}{1-a}$

[We hope this exercise suggests the following generalization:

$$\forall_n \forall_{a \neq 1} \frac{1}{1-a} = \sum_{p=1}^n a^{p-1} + \frac{a^n}{1-a}$$

From this it is easy to see that $\forall_n \forall_a 1 - a^n = (1-a) \sum_{p=1}^n a^{p-1}$.

See Theorem 153.]

18. (a) $x = \frac{cd}{bd - ae}$ (b) $x = \frac{ap + bq}{bp + aq}$ (c) $x = \frac{c - aq - bs}{ap - br}$

19. $K = a^2 - 2ab \left[a^2 - 4\frac{ab}{2} \right]$

20. (a) $x^2 - x - 42$ (b) $y^2 + y - 20$ (c) $c^2 + 13c + 42$

$$45. \$6000, \$4000 \quad [(3x - 1000)/(2x - 1000) = 5/3] \quad 46. 12$$

$$47. (a) x - 17y \quad [\text{or: } -8(x + y) + 9(x - y)]$$

$$(b) \frac{1}{2}(7m - 9n) \quad [\text{or: } 2(m - 3n) + \frac{3}{2}(m + n)]$$

$$48. X + Y + Z = X + \frac{C - A}{B - C}X + \frac{A - B}{B - C}X = X \left[\frac{(B - C) + (C - A) + (A - B)}{B - C} \right] = 0$$

$$[\text{Alternatively: } \frac{X}{B - C} = \frac{Y + Z}{(C - A) + (A - B)} = \frac{Y + Z}{C - B}, X = -(Y + Z)]$$

$$49. (a) 97/8$$

$$(b) 5$$

$$(c) 8$$

$$50. 52, 91 \text{ and } -91, -52 \quad [|4x - 7x| = 39]$$

So, since I^+ is closed with respect to addition and multiplication, there are positive integers m' and m'' such that $3(q+1)^5 + 5(q+1)^3 + 7(q+1) = 15m + 15m' = 15m''$.

Hence, $\frac{3(q+1)^5 + 5(q+1)^3 + 7(q+1)}{15} \in I^+$. Consequently,

$$\forall_n \left[\frac{3n^5 + 5n^3 + 7n}{15} \in I^+ \Rightarrow \frac{3(n+1)^5 + 5(n+1)^3 + 7(n+1)}{15} \in I^+ \right].$$

(iii) In view of (i) and (ii), it follows from the PMI that....

(b) Since $(1-2)(1-1)1(1+1)(1+2) = 0$, and $5 \mid 0$, and since

$$\begin{aligned} \forall_x (x-1)x(x+1)(x+2)(x+3) - (x-2)(x-1)x(x+1)(x+2) \\ = 5(x-1)x(x+1)(x+2), \end{aligned}$$

it follows by induction that

$$\forall_n 5 \mid (n-2)(n-1)n(n+1)(n+2).$$

Similarly,

$$\forall_n 3 \mid (n-1)n(n+1)$$

So, by the algebra theorem given in the exercise,

$$\forall_n 15 \mid 3n^5 + 5n^3 + 7n.$$

Hence,

$$\forall_n \frac{3n^5 + 5n^3 + 7n}{15} \in I^+.$$

$$40. \quad 15, 17 \text{ and } -17, -15 \quad [(2k-1)(2k+1) = 255 \iff (k=8 \text{ or } k=-8)]$$

$$41. \quad 7.5 \quad [2x - 5 = 10]$$

$$42. \quad (a) \quad \frac{c+d}{a-b}$$

$$(b) \quad \frac{a}{b}$$

$$(c) \quad \frac{2p-5q}{6p+7q}$$

$$43. \quad 9/4 \quad [a=3, b=5]$$

$$44. \quad 124, 4, 7 \quad [100p + 10l + w = 12697 - 250 = 100 \cdot 124 + 10 \cdot 4 + 7]$$

$$32. \quad 6, -1 \quad [k^2 + k = 6(k + 1) \iff (k = 6 \text{ or } k = -1)]$$

$$33. \quad (a) \text{ Yes. } [a_{43} = 43^2 - 3 \cdot 43 + 43 = 43(43 - 3 + 1)]$$

$$(b) \quad a_{42} \quad [\text{For your convenience, here is a list of the first 42 terms of } a: 41, 41, 43, 47, 53, 61, 71, 83, 97, 113, 131, 151, 173, 197, 223, 251, 281, 313, 347, 383, 421, 461, 503, 547, 593, 641, 691, 743, 797, 853, 911, 971, 1033, 1097, 1163, 1231, 1301, 1373, 1447, 1523, 1601, 1681]$$

[Call the attention of students to the table on page 8-250.] [For a discussion of prime-representing formulas, see page 8-210.]

$$34. \quad (a) \quad 56z^6 + 24z^5 + 32z^4 + 21z^3 + 58z^2 + 33z + 28$$

$$(b) \quad 4x^{10} + 20x^8 + 2x^7 + 25x^6 + 17x^5 + 36x^3 + 6x^2 + 15x + 3$$

$$35. \quad A + B + C = \frac{B-1}{B} + B + \frac{2}{B-1} = \frac{(B^2 - 2B + 1) + (B^3 - B^2) + 2B}{B(B-1)} = \frac{B^3 + 1}{B(B-1)}$$

$$36. \quad 50 \left[\frac{n(n-1)}{2} = 1225 \iff (n-50)(n+49) = 0 \right]$$

$$37. \quad (-1, 2.5, -0.5)$$

$$38. \quad V = V_0 + (i_1 + i_2 - e)t$$

$$39. \quad (a) \quad (i) \quad \frac{1}{5} + \frac{1}{3} + \frac{7}{15} = \frac{15}{15} = 1 \in I^+$$

(ii) Suppose that $\frac{3q^5 + 5q^3 + 7q}{15} \in I^+$. That is, suppose that there is a positive integer m such that $3q^5 + 5q^3 + 7q = 15m$. Now,

$$3(q+1)^5 + 5(q+1)^3 + 7(q+1)$$

$$= 3q^5 + 15q^4 + 35q^3 + 45q^2 + 37q + 15$$

$$= (3q^5 + 5q^3 + 7q) + 15(q^4 + 2q^3 + 3q^2 + 2q + 1).$$

$$6. \quad 3, \quad 3\left(1 + \frac{2}{1+1}\right), \quad 3\left(1 + \frac{2}{1+1}\right)\left(1 + \frac{2}{2+1}\right), \quad 3\left(1 + \frac{2}{1+1}\right)\left(1 + \frac{2}{2+1}\right)\left(1 + \frac{2}{3+1}\right);$$

$$4\left(1 + \frac{2}{1+1}\right)\left(1 + \frac{2}{2+1}\right)\left(1 + \frac{2}{3+1}\right)\left(1 + \frac{2}{4+1}\right)\left(1 + \frac{2}{5+1}\right)\left(1 + \frac{2}{6+1}\right)\left(1 + \frac{2}{7+1}\right)\left(1 + \frac{2}{8+1}\right)$$

*

Answers for Part B of the Exploration Exercises.

$$(3.) \quad 3^{100} \qquad (4.) \quad \frac{1}{101} \qquad (5.) \quad 10 \qquad (6.) \quad 5151 \quad [= \frac{101 \cdot 102}{2}]$$

... ..

51. at the end of the 7th minute

[The problem is nicely solved graphically. Otherwise, since the rate for the first 5 minutes is 3 gallons per minute and the effective rate during the first 9 minutes is $3\frac{4}{9}$ gallons per minute, we know that the rate was changed sometime after the end of the fifth minute. Suppose that it was changed at the end of x minutes. It follows that $3x + \frac{41 - 31}{11 - 9}(11 - x) = 41$. So, $x = 7$.]

52. (a) y^2

(b) $7x^2y$

(c) $-2b^3c^4$

(d) $7z^7$

*

Answers for Part A of the Exploration Exercises.

1. $5, 5 \cdot 1, 5 \cdot 1 \cdot 2, 5 \cdot 1 \cdot 2 \cdot 3; 5 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8$

2. $1, 1(1 + 1), 1(1 + 1)(2 + 1), 1(1 + 1)(2 + 1)(3 + 1);$

$1(1 + 1)(2 + 1)(3 + 1)(4 + 1)(5 + 1)(6 + 1)(7 + 1)(8 + 1)$

3. $3, 3 \cdot 3, 3 \cdot 3 \cdot 3, 3 \cdot 3 \cdot 3 \cdot 3; 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

4. $\frac{1}{2}, \frac{1}{2} \left(\frac{1+1}{1+2} \right), \frac{1}{2} \left(\frac{1+1}{1+2} \right) \left(\frac{2+1}{2+2} \right), \frac{1}{2} \left(\frac{1+1}{1+2} \right) \left(\frac{2+1}{2+2} \right) \left(\frac{3+1}{3+2} \right);$

$\frac{1}{2} \left(\frac{1+1}{1+2} \right) \left(\frac{2+1}{2+2} \right) \left(\frac{3+1}{3+2} \right) \left(\frac{4+1}{4+2} \right) \left(\frac{5+1}{5+2} \right) \left(\frac{6+1}{6+2} \right) \left(\frac{7+1}{7+2} \right) \left(\frac{8+1}{8+2} \right)$

5. $1, 1\sqrt{\frac{1+1}{1}}, 1\sqrt{\frac{1+1}{1}}\sqrt{\frac{2+1}{2}}, 1\sqrt{\frac{1+1}{1}}\sqrt{\frac{2+1}{2}}\sqrt{\frac{3+1}{3}};$

$1\sqrt{\frac{1+1}{1}}\sqrt{\frac{2+1}{2}}\sqrt{\frac{3+1}{3}}\sqrt{\frac{4+1}{4}}\sqrt{\frac{5+1}{5}}\sqrt{\frac{6+1}{6}}\sqrt{\frac{7+1}{7}}\sqrt{\frac{8+1}{8}}$

Read ' $\prod_{p=1}^n \dots$ ' as 'pi, from $p = 1$ to n of \dots ', or as 'the product of the

numbers \dots , for all $p \leq n$ '.

The recursive definition (*) is a way to remove the ambiguity from expressions of the form:

$$a_1 a_2 a_3 \dots a_p \dots a_n a_{n+1}$$

Note that in accord with our convention about doing multiplications from left to right, the expression:

$$\prod_{p=1}^n a_p \cdot a_{n+1}$$

is an abbreviation for:

$$\left(\prod_{p=1}^n a_p \right) a_{n+1} *$$

Answers for Part A.

$$1. \prod_{p=1}^4 p = \prod_{p=1}^3 p \cdot 4 = \prod_{p=1}^2 p \cdot 3 \cdot 4 = \prod_{p=1}^1 p \cdot 2 \cdot 3 \cdot 4 = 1 \cdot 2 \cdot 3 \cdot 4$$

$$2. (3 \cdot 1 - 1)^2$$

$$3. 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$$

$$4. \left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \left(1 + \frac{1}{4}\right) \left(1 + \frac{1}{5}\right) \left(1 + \frac{1}{6}\right) \left(1 + \frac{1}{7}\right) \left(1 + \frac{1}{8}\right)$$

[Students may rewrite this as:

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdot \frac{6}{5} \cdot \frac{7}{6} \cdot \frac{8}{7} \cdot \frac{9}{8}$$

and simplify to '9'. Thus, they will anticipate Exercise B1.]

The first of these is the fact that the
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Answers for Part B.

$$1. (i) \prod_{p=1}^1 \left(1 + \frac{1}{p}\right) = 1 + \frac{1}{1} = 2 = 1 + 1$$

$$\begin{aligned} (ii) \prod_{p=1}^{q+1} \left(1 + \frac{1}{p}\right) &= \prod_{p=1}^q \left(1 + \frac{1}{p}\right) \cdot \left(1 + \frac{1}{q+1}\right) \\ &= (q+1) \left(1 + \frac{1}{q+1}\right) \\ &= q + 1 + 1 \end{aligned}$$

(iii) So, in view of (i) and (ii), it follows by the PMI that...

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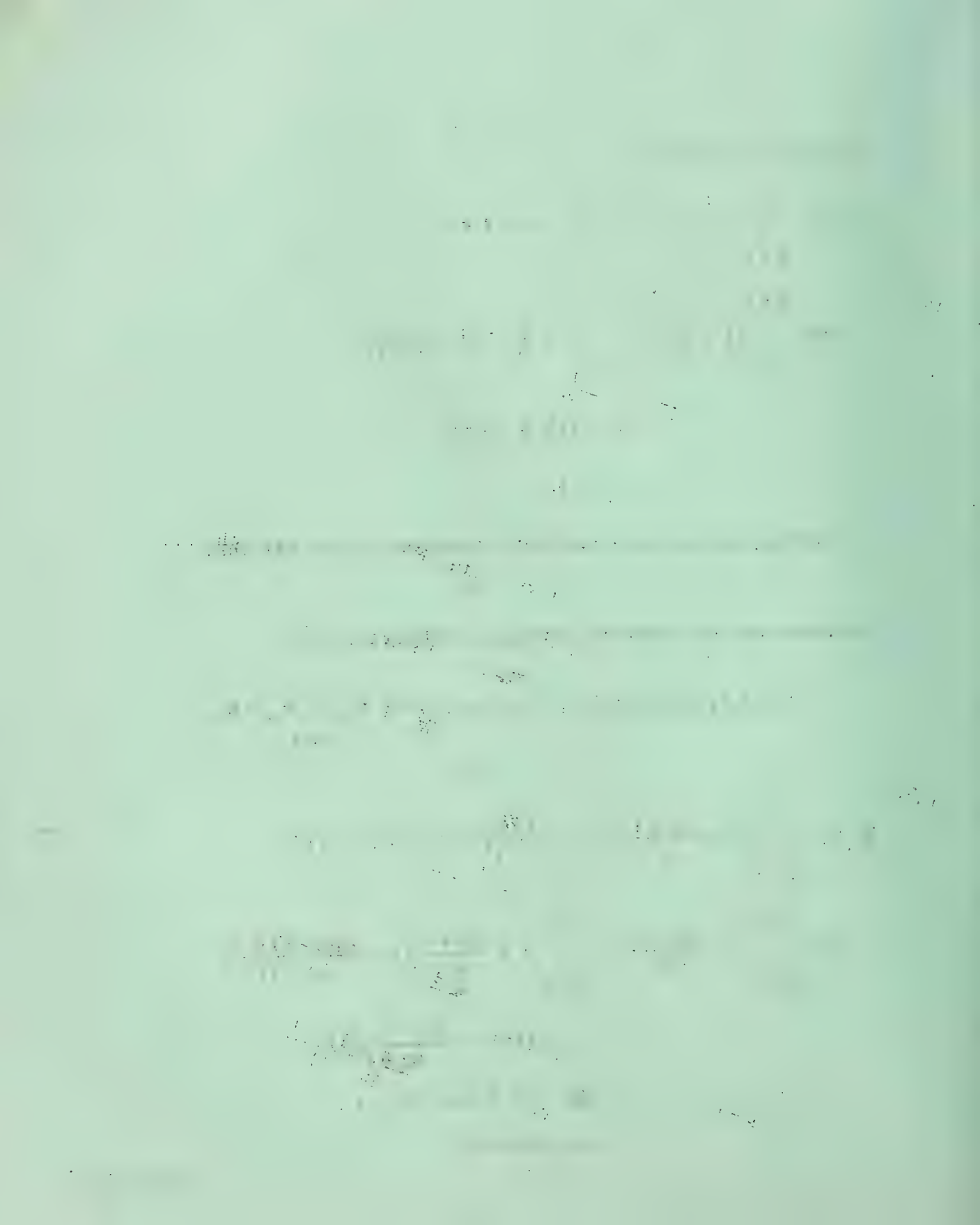
Students may discover an analogue of Theorem 130:

$$(b_1 = a_1 \text{ and } \forall_n b_{n+1} = b_n \cdot a_{n+1}) \Rightarrow \forall_n \prod_{p=1}^n a_p = b_n$$

*

$$2. (i) \prod_{p=1}^1 \left(1 + \frac{2p+1}{p^2}\right) = 1 + \frac{2 \cdot 1 + 1}{1^2} = 4 = (1 + 1)^2$$

$$\begin{aligned} (ii) \prod_{p=1}^{q+1} \left(1 + \frac{2p+1}{p^2}\right) &= \prod_{p=1}^q \left(1 + \frac{2p+1}{p^2}\right) \left(1 + \frac{2(q+1)+1}{(q+1)^2}\right) \\ &= (q+1)^2 \left(1 + \frac{2(q+1)+1}{(q+1)^2}\right) \\ &= (q+1)^2 + 2(q+1) + 1 \\ &= [(q+1) + 1]^2 \end{aligned}$$



(iii) So, ...

[The theorem in Exercise 2 is easy to discover if one notes the algebra theorem:

$$\forall_p \left(1 + \frac{2p+1}{p^2} \right) = \left(\frac{p+1}{p} \right)^2$$

and then simplifies a few instances. It should suggest an analogue of Theorem 138:

For any sequence a none of whose terms is 0,

$$\forall_n \left[\prod_{p=1}^n \frac{a_{p+1}}{a_p} = \frac{a_{n+1}}{a_1} \right]$$

$$5. \quad 1 \quad [= 1] \qquad 6. \quad 23 \quad [= (3 \cdot 7 + 2)] \qquad 7. \quad 1 \quad [= 1]$$

$$8. \quad 1 \quad [= 1] \qquad 9. \quad -36 \quad [= (-3 \cdot -2 \cdot -1)(1 \cdot 2 \cdot 3)]$$

*

Answers for Part B:

$$1. \quad 4 \qquad 2. \quad 4 \qquad 3. \quad 3 \quad [= 7 \cdot 0 + 3] \qquad 4. \quad 5 \qquad 5. \quad 0$$

$$6. \quad 20 \qquad 7. \quad 109 \quad [= 12^2 - 3 \cdot 12 + 1]$$

$$8. \quad 9701 \quad [= (11^2 - 3 \cdot 11 + 1) \cdot 109]$$

$$9. \quad 9 \quad [(2 \cdot 8 + 3)(2 \cdot 9 + 3) = 399]$$

$$10. \quad 4 \quad [\text{See Theorem 146 on page 8-99.}]$$

$$11. \quad 2 \quad [3 \cdot 1 + 7 = 10; \text{ See Th. 147.}]$$

$$12. \quad 539 \quad \left[\frac{(2 \cdot 1 + 9)(\cancel{2 \cdot 2 + 9}) \dots (\cancel{2 \cdot 19 + 9})(2 \cdot 20 + 9)}{(\cancel{2 \cdot 2 + 9}) \dots (\cancel{2 \cdot 19 + 9})} = 11 \cdot 49 \right]$$

$$13. \quad 6 \quad [1 = 1, 1 \cdot 2 = 2, 2 \cdot 3 = 6, 6 \cdot 4 = 24, 24 \cdot 5 = 120, 120 \cdot 6 = 720]$$

$$14. \quad 7 \quad [a_1 = 2, a_2 = 2, \dots; a_1 a_2 a_3 a_4 a_5 a_6 a_7 = 2^7 = 128]$$

$$15. \quad 6 \quad [a_0 = 2, a_1 = 2, \dots; a_0 a_1 a_2 a_3 a_4 a_5 a_6 = 2^7 = 128]$$

$$16. \quad 3 \quad [\text{See Th. 148.}]$$

$$17. \quad 14 \qquad 18. \quad 2$$

$$19. \quad 1 \quad \left[\left(1 + \frac{1}{p} \right) \left(1 - \frac{1}{p+1} \right) = 1 \right]$$

bottom of page 8-93:

$a_1 = x \cdot a_1$ if and only if $(x - 1)a_1 = 0$, and this is the case if and only if $x = 1$ or $a_1 = 0$.

*

The adoption of the convention according to which for any sequence a ,

$$\prod_{p=1}^0 a_p = 1$$

results in the same sort of simplifications as attend the adoption of the analogous convention for continued sums. In particular, it leads in a natural way [see pages 8-100 and 8-101] to the theorem:

$$\forall_x x^0 = 1$$

*

$$\prod_{i=-5}^4 i = 1 \cdot -5 \cdot -4 \cdot -3 \cdot -2 \cdot -1 \cdot 0 \cdot 1 \cdot 2 \cdot 3 \cdot 4 = 0$$

Note the leading '1·'. This results from application of the recursive definition. If, in evaluation problems such as this, one uses the theorem:

$$\forall_j \prod_{i=j}^j a_i = a_j$$

then such unnecessary '1·'s will not occur.

*

Answers for Part A.

1. 10395 [= 1·3·5·7·9·11]

2. 315 [= -3·-1·1·3·5·7]

3. 16 [= 2·2·2·2]

4. 16 [= 2·2·2·2]

3. (a) $(n + 1)! - 1$

$$\left[\sum_{p=1}^q p \cdot p! = \sum_{p=1}^q [(p+1)! - p!] = (q+1)! - 1 \right]$$

(b) $1 - \frac{1}{(n+1)!}$

$$\left[\sum_{p=1}^q \frac{p}{(p+1)!} = \sum_{p=1}^q \left[\frac{1}{(p+1)!} - \frac{1}{p!} \right] = 1 - \frac{1}{(q+1)!} \right]$$

4. (a) (i) $1! = 1 \geq 1$

(ii) Suppose that $p! \geq p$. Then, $p! \geq 1$ and, since $p+1 \geq 0$, it follows that $p!(p+1) \geq p+1$. So, by the recursive definition for the factorial sequence, $(p+1)! \geq p+1$. Etc.

(b) (i) $\sum_{k=0}^1 \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} = 2 \leq 3 - \frac{1}{1}$

(ii) $\sum_{k=0}^{q+1} \frac{1}{k!} = \sum_{k=0}^q \frac{1}{k!} + \frac{1}{(q+1)!}$

$$\leq 3 - \frac{1}{q!} + \frac{1}{(q+1)!}$$

$$= 3 - \frac{q}{(q+1)!}$$

$$\leq 3 - \frac{1}{(q+1)!}$$

[See discussion of Exercise ☆4 of Part B on page 8-12.]

[You may recall that the case, e , of Napierian logarithms is given by:

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

Exercise 4(b) gives a way of showing that $e \leq 3$. More basically, it shows that the summation sequence of the sequence of reciprocal factorials is bounded.]

$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x) e^{-x^2} dx = \frac{1}{\sqrt{\pi}}$

1. *Chrysomelidae* (Coleoptera) (100%)

.....

$$(i) \sum_{i=0}^0 g_i = g_0 = 1 = g_1 - 1$$

$$(ii) \sum_{i=0}^q g_i = \sum_{i=0}^{q-1} g_i + g_q = (g_q - 1) + g_q = \underbrace{2g_q - 1}_{\text{def. of } \pi\text{-notation}} = g_{q+1} - 1$$

4. 9 [This is a straight-forward computing problem which foreshadows the factorial sequence treated on the next page in the text.]

| k | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|---|---|----|-----|-----|------|-------|--------|---------|
| g_k | 1 | 1 | 2 | 6 | 24 | 120 | 720 | 5040 | 40320 | 362880 | 3628800 |

Since $g_9 < 10^6$, $g_{10} > 10^6$, and $\forall_n g_{n+1} \geq g_n$ [definition of g_n and Theorem 104], it follows that 9 is the smallest integer m such that for all $n > m$, $g_n > 10^6$.]

*

Answers for Part D.

1. See the discussion of Exercise 4 of Part C, above.

2. (a) $20 \left[= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3} \right]$

(b) 12

(c) 6

(d) $\frac{1}{17160} \left[= \frac{1}{10 \cdot 11 \cdot 12 \cdot 13} \right]$

(e) $\frac{1}{40320} \left[= \frac{1}{8!} \right]$

(f) 3003

(g) $\frac{1}{288}$

(h) 120

(i) 92378

(j) 210

2. 99

[Computing a few of the values of f will lead students to discover the theorem:

$$\forall_n f_n = \frac{1}{n+1}$$

which should be proved by mathematical induction.

$$\left[\begin{array}{l} \text{(i) } f_1 = \prod_{p=1}^1 \frac{p}{p+1} = \frac{1}{1+1} \\ \text{(ii) } f_{q+1} = \prod_{p=1}^{q+1} \frac{p}{p+1} = \prod_{p=1}^q \frac{p}{p+1} \cdot \frac{q+1}{q+2} = \frac{1}{q+1} \cdot \frac{q+1}{q+2} = \frac{1}{q+2} \end{array} \right]$$

$$\text{Then, } f_n < 0.01 \iff \frac{1}{n+1} < 0.01$$

$$\iff n+1 > 100$$

$$\iff n > 99.$$

Students should be asked to find an n such that $f_n = 0$. Of course, there is none.]

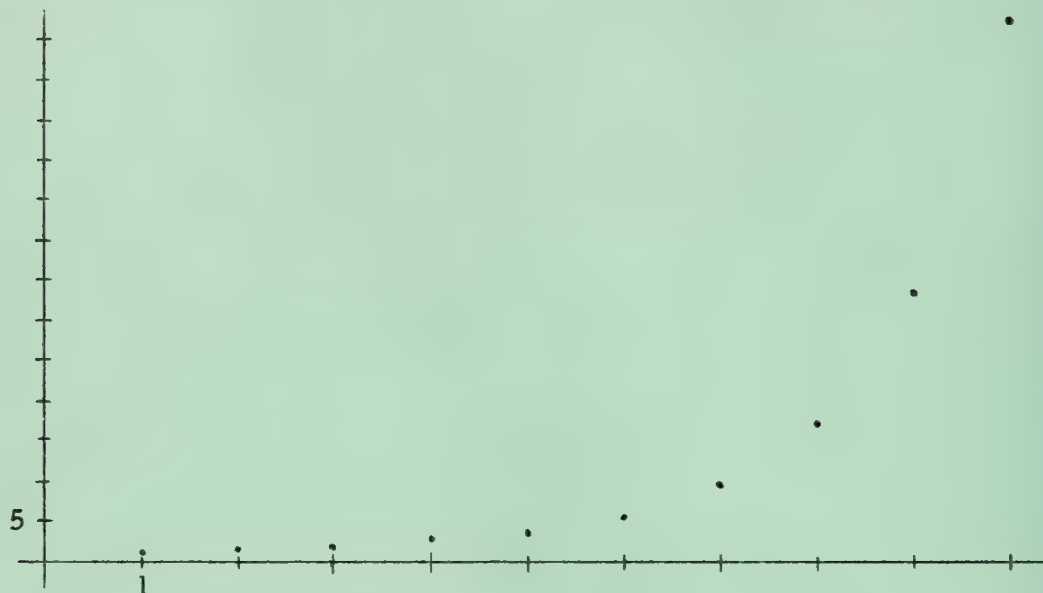
3. (a)

| k | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|--------------------|---|---|---|----|----|----|-----|-----|
| g_k | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 |
| $\sum_{i=0}^k g_i$ | 1 | 3 | 7 | 15 | 31 | 63 | 127 | 255 |

$$\text{(b) } \forall_n \sum_{i=0}^{n-1} g_i = g_n - 1$$

Answers for Part C.

1.



For your convenience, here is a table which gives the relevant values.

| p | $1 + \frac{p}{10}$ | n | f_n |
|----|--------------------|----|-------------|
| 1 | 1.1 | 1 | 1.1 |
| 2 | 1.2 | 2 | 1.32 |
| 3 | 1.3 | 3 | 1.716 |
| 4 | 1.4 | 4 | 2.4024 |
| 5 | 1.5 | 5 | 3.6036 |
| 6 | 1.6 | 6 | 5.76576 |
| 7 | 1.7 | 7 | 9.801792 |
| 8 | 1.8 | 8 | 17.6432256 |
| 9 | 1.9 | 9 | 33.52212864 |
| 10 | 2 | 10 | 67.04425728 |

[Theorem 133 may suggest another generalization to some of your students:

$$\forall_x \forall_j \forall_{k \geq j-1} \sqrt[k]{\prod_{i=j}^k x^{a_i}} = x$$

If the terms of the sequence a are restricted to be nonnegative, this is a generalization of the theorem of Exercise 2 of Part B on page 8-101. With the further restriction that the terms of a are integers it can easily be proved by induction. If the quantifier ' \forall_x ' is replaced by ' $\forall_x \neq 0$ ', it is a generalization of Theorem 155 and can be proved under the restriction that the terms of a are integers. After Unit 9, it can be proved with no restrictions on the sequence a , if ' \forall_x ' is replaced by ' $\forall_x > 0$ '.]

*

Note that exponentiation is, like addition, a binary operation. [Actually, exponentiation bears somewhat more resemblance to division than to addition in that, like division, it is not defined for all pairs of real numbers--for example, for the pair $(-1, 1/2)$.] Note also that questions such as 'Is 81 a power?' are as trivial as 'Is 15 a sum?' or 'Is 5 a quotient?'. The answer to all such questions is 'yes'. One can, more or less reasonably, ask:

What is the divisor of the quotient 15 if the dividend is 3?,

What is the first summand of the sum 5 if the second summand is 7?,

What is the exponent of the power 81 if the base is 3?,

but it is less confusing to say: Solve ' $15 = 3/x$ ', ' $5 = x + 7$ ', ' $81 = 3^x$ '.

The proofs of Theorems 145-149, mostly by mathematical induction, are straightforward adaptations of those of Theorems 134-137 and Theorem 142, respectively. For example, consider the proof of Theorem 134 given in the COMMENTARY for Part D on page 8-43. If, in this proof, one replaces ' Σ 's by ' Π 's, '+'s by ' \cdot 's, and '0's by '1's, he has a proof of Theorem 145. [In the explanation of the proof of Theorem 134, one should, also, replace the reference to the sum rearrangement theorem--Theorem 5--by a reference to the product rearrangement theorem--Theorem 4.] The proofs of Theorems 135 and 137 given in the COMMENTARY for Part $\star F$ on page 8-47 can be modified in the same way to yield proofs of Theorems 146 and 148. Theorem 147 is a corollary of Theorem 146, just as Theorem 136 is a corollary of Theorem 135 [See, again, the discussion of Part $\star F$ on page 8-47.]. Finally, Theorem 149 can be proved just as Theorem 142 is proved in the COMMENTARY for pages 8-73 and 8-74. The latter proof makes use of the cpa and Theorems 136 and 137. The proof of Theorem 149 makes use of the cpm and Theorems 147 and 148. [Descriptive names are given in the Table of Contents on page 8-viii.]

*

Π - theorems analogous to Theorems 130 and 138 have been noted in the COMMENTARY for Part B on page 8-92. Once students, on studying page 8-100, discover the analogue:

$$(\Pi) \quad \forall_x \forall_{k \geq 0} \overline{\prod_{p=1}^k} x = x^k$$

of the given theorem (Σ) [The missing word in line 9 is 'power'], they may suggest the analogue:

$$\forall_x \forall_j \forall_{k \geq j-1} \overline{\prod_{i=j}^k} a_i^x = \left(\overline{\prod_{i=j}^k} a_i \right)^x$$

of Theorem 133. This is a generalization of Theorem 158 [page 8-119]. With 'x' replaced by ' k_1 ' and with the restriction that the terms of the sequence a are all different from 0, this generalization can be proved just as Theorem 133 is proved in the COMMENTARY for pages 8-39 and 8-40. [Instead of the ldpma one must use Theorem 158.] [Once the definition of exponentiation has been completed, in Unit 9, the same proof will go through without 'x' being replaced by ' k_1 ', if the terms of a are further restricted to be positive.]

$$\begin{aligned}
2. \quad c^{j+k} &= \overbrace{\prod_{p=1}^{j+k}} c && [\text{definition; } (j \geq 0 \text{ and } k \geq 0) \Rightarrow j+k \geq 0] \\
&= \overbrace{\prod_{p=1}^j} c \cdot \overbrace{\prod_{p=j+1}^{j+k}} c && [\text{Theorem 146; } j \geq 1-1; k \geq 0 \Rightarrow j+k \geq j] \\
&= \overbrace{\prod_{p=1}^j} c \cdot \overbrace{\prod_{p=1}^k} c && [\text{Theorem 148; } k \geq 1-1] \\
&= c^j \cdot c^k && [\text{definition; } j \geq 0; k \geq 0]
\end{aligned}$$

Now, workers in cardinal number theory invariably define exponentiation in a very simple and natural manner which has as one of its consequences that, for cardinal numbers, $0^0 = 1$. So, if one is to maintain the imaging of each number system in the succeeding one, one must accept the consequence that, in any of the successive number systems including the system of real numbers, $0^0 = 1$. [Note, still that the definition, for real numbers, of ' 0^0 ' is only a matter of convenience. One could give up the imaging. But, this would be inconvenient.]

*

Answers for Part A.

- | | | | |
|--------------------------------|----------|---------|-------------|
| 1. 1 | 2. 1 | 3. 1 | 4. 1 |
| 5. 1 | 6. 1 | 7. 0 | 8. 1 |
| 9. 1024 | 10. 4096 | 11. 256 | 12. 1048576 |
| 13. -1 | 14. 1 | 15. 243 | 16. -243 |
| 17. 19683 | 18. 324 | 19. 0 | |
| 20. 39366 [= $3^9 (3^1 - 1)$] | | | |

*

Answers for Part B. [Old friends are finally proved!]

$$1. (cd)^k = \prod_{p=1}^k (cd) = \prod_{p=1}^k c \cdot \prod_{p=1}^k d = c^k d^k$$

\swarrow def. \swarrow Theorem 145 \swarrow def.

[Each step requires the restriction ' $k \geq 0$ '.]

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• •

definition can have is convenience. Is it convenient to say that $0^0 = 1$? The greatest mass of evidence that it is convenient to do so is contained in the advanced mathematics texts [from calculus on] which adopt this convention. [If such books make any explicit reference to ' 0^0 ', they usually say that it is not defined. So, their adoption of the convention that $0^0 = 1$ is usually implicit--but it's there.] All one need do is turn to any treatment of infinite series to see many statements similar to:

$$\sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}, \text{ for } |x| < 1 \quad [\text{cf. Theorem 168b}]$$

Since $|0| < 1$, one instance of this generalization is:

$$\sum_{n=1}^{\infty} 0^{n-1} = \frac{1}{1-0},$$

which, for our present purposes, may be paraphrased as:

$$0^0 + 0^1 + 0^2 + \dots + 0^{n-1} + \dots = 1$$

Clearly, this is meaningless unless ' 0^0 ' is defined, and is false unless $0^0 = 1$. [Oddly enough, the same writers take considerable pains to point out that, by definition, $0! = 1$, just for the purpose of making sense out of similar theorems, such as:

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = e^x, \text{ for all } x \quad]$$

From the mass of textbook writers one should single out one prominent mathematician, Landau, who, in his Differential and Integral Calculus [Chelsea 1951, page 11] says:

Let 0^0 always be understood as meaning 1.

There is still another reason why it is desirable to adopt the definition ' $0^0 = 1$ '. As you are aware, one kind of procedure for defining the real numbers begins with the cardinal numbers and, in terms of these defines numbers of another kind [integers or, alternatively, unsigned rationals], in terms of which still others are defined, in terms of which are, finally, defined the real numbers. Concomitantly with defining different kinds of numbers, one defines operation on them--at least operations of addition and multiplication. The procedure is such that each number system has its analogue in the succeeding one--technically, each number system has an isomorphic image in the succeeding one.

presented by calculus is that of convincing students of the illegitimacy of such a substitution procedure for finding limiting values, this last suggestion is a particularly unfortunate one.

Summarizing the discussion thus far, it appears that the misleading character of the phrase 'the indeterminate form 0^0 ' is enough to make it an insecure basis for an argument against defining ' 0^0 ' to be a numeral.

Proceeding, now, to the argument against defining ' 0^0 ' which is found in most elementary algebra texts, we recall that it develops through the attempt to motivate the definition of expressions such as ' 2^{-3} '.

Since, for any positive integers m and $n < m$, $2^m/2^n = 2^{m-n}$, it seems desirable to define ' 2^{-3} ', say, so that, for example, $2^2/2^5 = 2^{-3}$. If this is to be the case, it is evident that ' 2^{-3} ' must be a numeral for $1/2^3$. On adopting the definition which this suggests, it is not hard to prove that, for any integers j and $k \neq j$, $2^j/2^k = 2^{j-k}$. Now, one considers the case $j = k$. Since, for any k , $2^k/2^k = 1$, it is natural to accept ' 2^0 ' as a numeral for 1. But--let's be careful. Whatever k is, ' $0^k/0^k$ ' is nonsense. So, ' 0^{k-k} ' must be nonsense also! You mustn't define ' 0^0 '.

This is, course, an unjustified conclusion. Whether ' 0^0 ' is defined or not [and, if it is, how it is defined] has no relevance for the theorem one wishes to prove:

$$\forall_{x \neq 0} \forall_j \forall_k x^j/x^k = x^{j-k} \quad [\text{Theorem 156}]$$

The fact that ' $0^k/0^k$ ' is nonsense merely precludes the possibility of motivating the definition ' $0^0 = 1$ ' in the way ' $2^0 = 1$ ' was motivated.

So far we have treated the objections to defining ' 0^0 '. Now, let's consider the advantages of the definition ' $0^0 = 1$ '. The only advantage any

As a matter of fact, if one knows something about logarithms, including the theorem:

$$\forall_{x > 0} \forall_y \log x^y = y \log x$$

and the fact that $|\log x|$ is arbitrarily large if x is a sufficiently small positive number, it is very easy to prove (**). In fact, one can prove a much stranger result--that

no matter what number $z > 0$ one chooses other than
 (***) 1, there are nonzero numbers x and y , as near 0 as
 one wishes, such that $x^y = z$.

For, given a number $c > 0$, $\log c \neq 0$ and, for $a > 0$, $a^b = c$ if and only if $b = \log c / \log a$. Since, for a sufficiently near 0, $|\log a|$ is arbitrarily large, it follows that, for a sufficiently near 0, $\log c / \log a$ will be different from 0 but as near 0 as one wishes. So, there are nonzero numbers a and b , as near 0 as one wishes, such that $a^b = c$.

Expressions which, like $x^{\frac{\log c}{\log x}}$, are of the form:

$$f(x)g(x)$$

--when f and g are functions whose values, for arguments near 0, are also, near 0--are said to be 'of the indeterminate form 0^0 '. Among the techniques of the differential calculus there is one which, in certain cases, enables one to discover how the values of a given such expression behave for small nonzero values of ' x '. When [as in the case treated above] these values cluster about some number $[c]$, the expressions are not, in the intended sense, indeterminate. In any case, speaking of such expressions as being "of the form 0^0 " is misleading because it suggests --what is often not the case--that $f(0) = 0 = g(0)$ and that the values of the expression for small nonzero values of ' x ' do cluster about some number $f(0)g(0)$. Since one of the most difficult pedagogical problems

Objections are often raised against defining ' 0^0 '. The basis of these objections has to do with the nature of the exponential function [$z = x^y$] when it is defined [as is done in Unit 9] for positive bases and arbitrary real exponents. It is then the case that

- (*) no matter what number z one chooses [as a referent for ' 0^0 '] there are nonzero numbers x and y , as near 0 as one wishes, such that x^y is not near z .

[In fact, there are such numbers x and y for which $|x^y - z|$ is as large as one may wish.] In more technical language, the exponential function does not have a limit at $(0, 0)$. As a consequence of this, it is impossible to define ' 0^0 ' in such a way that the exponential function [which would, then, be defined at $(0, 0)$ as well as on $\{(x, y): x > 0\}$] would be continuous at $(0, 0)$.

Note that the property of exponentiation just described does not, in any way, preclude one from defining ' 0^0 '. Its only effect is that, no matter how one does this, exponentiation will fail to have a certain desirable property. The question of whether it is desirable to define ' 0^0 ' is left wide-open.

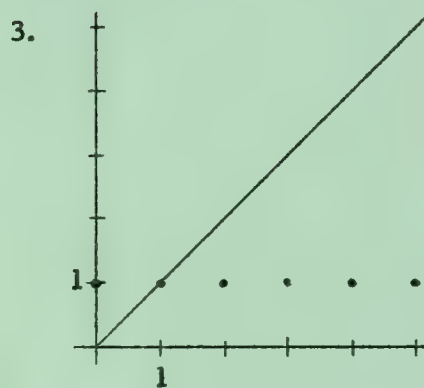
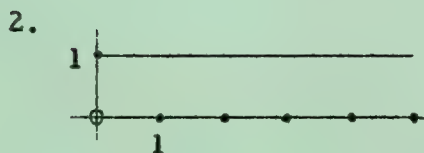
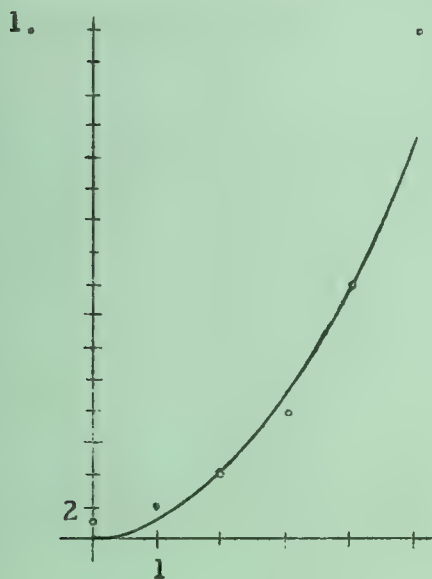
Now, it cannot rationally be argued, merely in the ground of the fact mentioned above, that ' 0^0 ' should not be defined. For there are many useful functions which fail to be continuous at certain places where they are defined [the greatest integer function is one]. Instead, the only reputable argument against defining ' 0^0 ' rests on the fact that it is customary, in calculus texts, to use the phrase 'the indeterminate form 0^0 ' as a section-heading under which to discuss problems which arise in consequence of another theorem, stranger than (*), about the behavior of exponentiation for "small" bases and exponents. [The disreputable argument against defining ' 0^0 ' which is commonly given in elementary algebra texts will be discussed later.]

According to the theorem, stronger than (*), just referred to, it is the case that

- (**) no matter what number $z \geq 0$ one chooses there are nonzero numbers x and y , as near 0 as one wishes, such that $|x^y - z|$ is as small as one may wish.

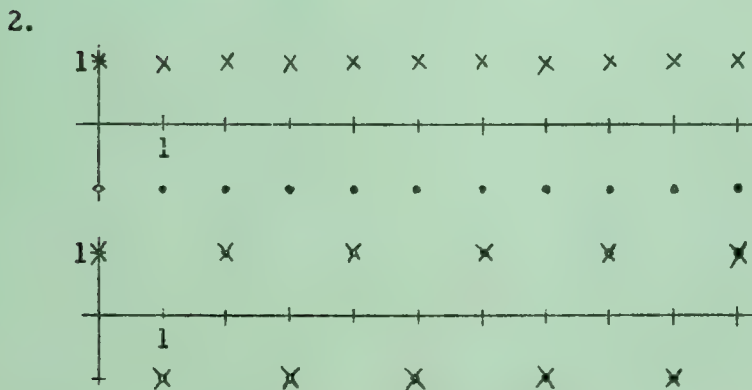
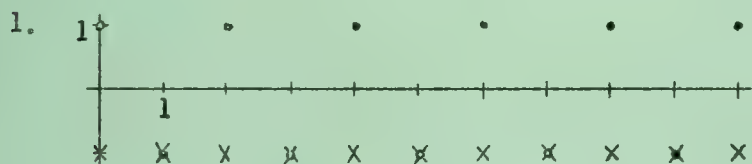
[It may seem at first that (**) contradicts (*), but, actually, (**) implies (*). For, values of ' x^y ' which are as close as one wishes to a given nonnegative number are not near to "most" numbers other than the given number. For example, to find values of ' x^y ' which are not near to 1, it is sufficient to find values which are near to 1000.]

Answers for Part C.



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Answers for Part D.



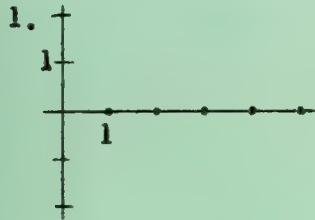
[Students should repeat Exercise 1 of Part D with '1' replaced by '2'.]

1. The first part of the paper is devoted to a general discussion of the problem of the existence of solutions of the system of equations (1) for arbitrary values of the parameters α and β . It is shown that the system (1) has solutions for arbitrary values of the parameters α and β if and only if the condition $\alpha + \beta = 1$ is satisfied. In this case the solutions are unique and are given by the formulas (2).

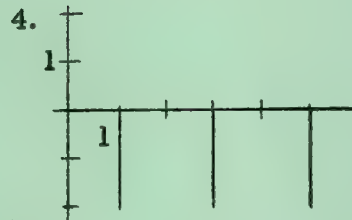
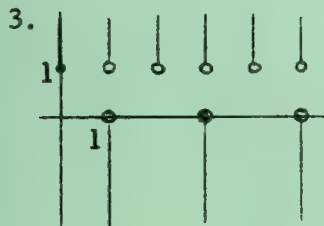
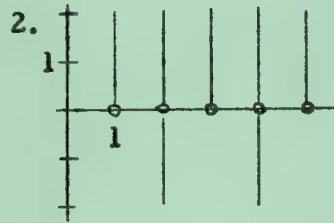
2. The second part of the paper is devoted to a study of the properties of the solutions of the system (1) for arbitrary values of the parameters α and β . It is shown that the solutions of the system (1) are continuous functions of the parameters α and β and that they are differentiable with respect to the parameters α and β if the condition $\alpha + \beta = 1$ is satisfied.

3. The third part of the paper is devoted to a study of the properties of the solutions of the system (1) for arbitrary values of the parameters α and β . It is shown that the solutions of the system (1) are continuous functions of the parameters α and β and that they are differentiable with respect to the parameters α and β if the condition $\alpha + \beta = 1$ is satisfied. In this case the solutions are unique and are given by the formulas (2).

Answers for Part E.

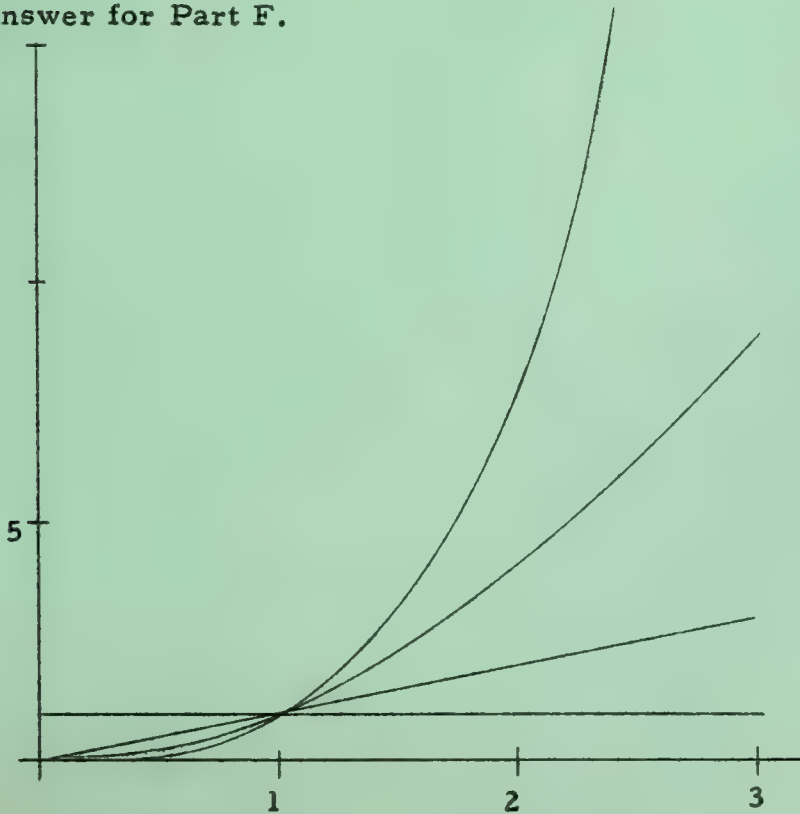


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Answer for Part F.



20

0.0125 0.0125 0.0125



20



0.0125 0.0125 0.0125



FIG. 1. A schematic diagram of a mechanical system, showing a horizontal beam supported by a vertical column. The beam is labeled with 'A' and 'B' at its ends, and the column is labeled with 'C' at its base. The diagram is oriented vertically on the page.

FIG. 2. A schematic diagram of a mechanical system, showing a horizontal beam supported by a vertical column. The beam is labeled with 'A' and 'B' at its ends, and the column is labeled with 'C' at its base. The diagram is oriented vertically on the page.

FIG. 3. A schematic diagram of a mechanical system, showing a horizontal beam supported by a vertical column. The beam is labeled with 'A' and 'B' at its ends, and the column is labeled with 'C' at its base. The diagram is oriented vertically on the page.

FIG. 4. A schematic diagram of a mechanical system, showing a horizontal beam supported by a vertical column. The beam is labeled with 'A' and 'B' at its ends, and the column is labeled with 'C' at its base. The diagram is oriented vertically on the page.

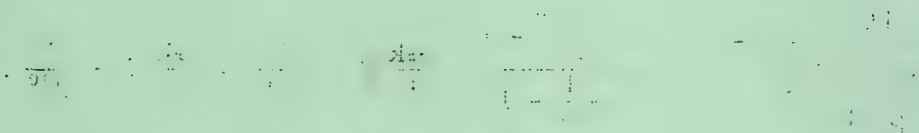


FIG. 5. A schematic diagram of a mechanical system, showing a horizontal beam supported by a vertical column. The beam is labeled with 'A' and 'B' at its ends, and the column is labeled with 'C' at its base. The diagram is oriented vertically on the page.

FIG. 6. A schematic diagram of a mechanical system, showing a horizontal beam supported by a vertical column. The beam is labeled with 'A' and 'B' at its ends, and the column is labeled with 'C' at its base. The diagram is oriented vertically on the page.



FIG. 7. A schematic diagram of a mechanical system, showing a horizontal beam supported by a vertical column. The beam is labeled with 'A' and 'B' at its ends, and the column is labeled with 'C' at its base. The diagram is oriented vertically on the page.

FIG. 8. A schematic diagram of a mechanical system, showing a horizontal beam supported by a vertical column. The beam is labeled with 'A' and 'B' at its ends, and the column is labeled with 'C' at its base. The diagram is oriented vertically on the page.

$$k \sum_{p=1}^{10} \left(-\frac{1}{4}\right)^{p-1}$$

and the second can be rewritten as:

$$\frac{3k}{4} \sum_{p=1}^5 \left(\frac{1}{16}\right)^{p-1}$$

Simplifying the first of the two expressions, we see that

$$k \sum_{p=1}^{10} \left(-\frac{1}{4}\right)^{p-1} = k \frac{\left(-\frac{1}{4}\right)^{10} - 1}{-\frac{1}{4} - 1} = \frac{4k}{5} \left[1 - \frac{1}{4^{10}}\right] = \frac{4k}{5} \left[1 - \frac{1}{2^{20}}\right].$$

Since $2^{20} = 1048576$, $1 - (1/2^{20}) \doteq 0.999999$, and the sum of the area-measures of the shaded regions is approximately $0.799999k$. [It is precisely $0.799999237060546875k$.]

Simplifying the second expression given above, we see that

$$\frac{3k}{4} \sum_{p=1}^5 \left(\frac{1}{16}\right)^{p-1} = \frac{3k}{4} \frac{\left(\frac{1}{16}\right)^5 - 1}{\frac{1}{16} - 1} = \frac{4k}{5} \left[1 - \frac{1}{16^5}\right].$$

[Some students should wonder about "the sum of the area-measures of all the shaded triangles". This leads into the subject of infinite geometric progressions, which is taken up on pages 8-138 ff. The question of what "the sum of infinitely many numbers" means is a subtle one, and it would be well, at this time, to answer an inquiry about the sum of all the area-measures by saying 'Well, at any rate, it's clear that the more triangles you take into consideration the closer the sum of the area-measures will be to $0.8k$.']

Answers for the rest of Part H are in the COMMENTARY for pages 8-105 and 8-106.

$\frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2}$

$$\frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2}$$

...

...

... (d) ...
 ...
 ...
 ...
 ...
 ...

...

...

...

...

$$\frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2}$$

...

...

...

...

$$\begin{aligned}
& 2^9 + 2^8 + \dots + 2^2 + 2^1 + 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots + \left(\frac{1}{2}\right)^{10} \\
&= \frac{1}{2^{10}} [2^{19} + 2^{18} + \dots + 2^{12} + 2^{11} + 2^{10} + 2^9 + 2^8 + \dots + 2^0] \\
&= \frac{1}{2^{10}} \sum_{p=1}^{20} 2^{p-1} = \frac{1}{2^{10}} \cdot \frac{2^{20} - 1}{2 - 1} = 2^{10} - \frac{1}{2^{10}} \\
&= 1024 - 0.0009765625 \\
&\doteq 1024.
\end{aligned}$$

(b) The phrase 'the shaded regions of the first ten triangles' should suggest the three shaded regions at the corners of the first triangle, the three at the corners of the third triangle, and so on, ending with the three at the corners of the ninth triangle. For each n , the area-measure of the n th triangle is $k/4^{n-1}$. If n is odd, the sum of the area-measures of the three shaded regions at the corners of the n th triangle is, by "subtraction of areas",

$$k/4^{n-1} - k/4^n,$$

or, by "addition of areas",

$$3 \cdot \frac{k}{4^n}.$$

So, the sum of the area-measures can be computed by simplifying either:

$$\left(\frac{k}{4^0} - \frac{k}{4^1}\right) + \left(\frac{k}{4^2} - \frac{k}{4^3}\right) + \dots + \left(\frac{k}{4^8} - \frac{k}{4^9}\right)$$

or:

$$\frac{3k}{4^1} + \frac{3k}{4^3} + \dots + \frac{3k}{4^9}$$

The first expression can be rewritten as:

$$(e) \ 1016 \quad \left[\sum_{p=4}^{10} 2^{p-1} = \sum_{p=1}^7 2^{p-1+3} = 8 \sum_{p=1}^7 2^{p-1} \right] \quad (f) \ 1$$

$$(g) \ 97655 \quad \left[\sum_{p=1}^7 5^p = \sum_{p=2}^8 5^{p-1} = \sum_{p=1}^8 5^{p-1} - 5^0 \right]$$

(h) 10 [The instance for $x = 1$ of Theorem 153 is not helpful.
Use Theorems 150a and 131.]

3. [Here is the problem:

Stan Brown has a choice between two jobs. Each job is to last for 30 days. The first job pays \$100 per day. The second job starts with 1 cent the first day, 2 cents the second day, 4 cents the third day, etc. doubling each day. Which job pays more? By how much?]

The first job pays $\sum_{p=1}^{30} 100$ dollars--that is, \$3000.

The second job pays $\sum_{p=1}^{30} 2^{p-1}$ cents--that is, 1073741823 cents.

So, the second job pays \$10734418.23 more than the first.

4. (a) For each n , the perimeter of the n th triangle is twice the perimeter of the $(n + 1)$ th triangle. So, the sum of the perimeters of the first twenty triangles is

$$f(x) = \frac{1}{2} \log \frac{1+x}{1-x}$$

Let $f(x) = \frac{1}{2} \log \frac{1+x}{1-x}$. Then $f'(x) = \frac{1}{1-x^2}$. We have $f(0) = 0$ and $f(1) = \frac{1}{2} \log 2$.

$$f(x) = \frac{1}{2} \log \frac{1+x}{1-x}$$

$$f'(x) = \frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)}$$

$$f(x) = \frac{1}{2} \log \frac{1+x}{1-x}$$

$$f'(x) = \frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)}$$

$$f(x) = \frac{1}{2} \log \frac{1+x}{1-x}$$

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$$f(x) = \frac{1}{2} \log \frac{1+x}{1-x}$$

6. (i) By the r.d., $c^0 = 1 \neq 0$ [basic principle].

(ii) Suppose that $c^j \neq 0$. By the cancellation principle for multiplication, it follows, for $c \neq 0$, that $c^j \cdot c \neq 0 \cdot c = 0$. Since...
[continue as in answer to Exercise 5].

There is a longer proof of Theorem 152b which makes use of Theorem 152a. It is of some pedagogical interest at this point since it also makes use of the following generalization of Theorem 150c:

$$\forall_x \forall_{k \geq 0} [(-x)^{2k} = x^{2k} \text{ and } (-x)^{2k+1} = -x^{2k+1}]$$

Here is this proof of Theorem 152b:

Suppose that $c > 0$. Then, by Theorem 152a, $c^j > 0$ and, by Theorem 81, $c^j \neq 0$. Hence, if $c > 0$ then $c^j \neq 0$.

Suppose that $c < 0$. Then, by Theorem 85, $-c > 0$ and, as above, $(-c)^j \neq 0$. By the theorem in Exercise 1 of Part B [and algebra], $(-c)^j = (-1)^j c^j$. Now, if j is even, it follows by Theorem 150c that $(-c)^j = c^j$ and, if j is odd, it follows that $(-c)^j = -c^j$. In either case, since $(-c)^j \neq 0$, it follows that $c^j \neq 0$. Since j is either even or odd, it follows that if $c < 0$ then $c^j \neq 0$.

Since, for $c \neq 0$, either $c > 0$ or $c < 0$, and since if $c > 0$ then $c^j \neq 0$ and if $c < 0$ then $c^j \neq 0$, it follows that, for $c \neq 0$, $c^j \neq 0$.

Consequently, $\forall_{x \neq 0} \forall_{k \geq 0} x^k \neq 0$.

7. For $c > 1$, $c > 0$ and, by Theorem 152a, $c^j > 0$. Hence, by the ntpi [and the cpm], $c^j \cdot c > c^j \cdot 1 = c^j$. Since $c^j \cdot c = c^{j+1}$, it follows, for $c > 1$, that $c^{j+1} > c^j$. Consequently, $\forall_{x > 1} \forall_{k \geq 0} x^{k+1} > x^k$

*

Consequently,

$$\begin{aligned} &\text{if } (-1)^{2j} = 1 \text{ and } (-1)^{2j+1} = -1 \text{ then} \\ &(-1)^{2(j+1)} = 1 \text{ and } (-1)^{2(j+1)+1} = -1. \end{aligned}$$

(iii) From (i) and (ii) it follows, by Theorem 114, that

3. By the r.d., $0^0 = 1$.

(i) By the r.d., $0^1 = 0^{0+1} = 0^0 \cdot 0 = 1 \cdot 0 = 0$.

(ii) Suppose that $0^P = 0$. By the r.d., $0^{P+1} = 0^P \cdot 0 = 0 \cdot 0$.
Consequently, $\forall_n [0^n = 0 \Rightarrow 0^{n+1} = 0]$.

(iii) From (i) and (ii) it follows, by the PMI, that $\forall_n 0^n = 0$.

Consequently, $0^0 = 1$ and $\forall_n 0^n = 0$.

4. (i) By the r.d., $p^0 = 1$ and, by (I_1^+) , $1 \in I^+$. Hence, $p^0 \in I^+$.

(ii) Suppose that $p^j \in I^+$. By the r.d., $p^{j+1} = p^j \cdot p$ and, by Theorem 103 and the inductive hypothesis, $p^j \cdot p \in I^+$. Hence, if $p^j \in I^+$ then $p^{j+1} \in I^+$.

(iii) From (i) and (ii) it follows, by Theorem 114, that $\forall_{k \geq 0} p^k \in I^+$.

Consequently, $\forall_m \forall_{k \geq 0} m^k \in I^+$.

[Theorem 151b is given in Exercise 1 of Part A on page 8-107.]

5. (i) By the r.d., $c^0 = 1 > 0$ [Theorem 82].

(ii) Suppose that $c^j > 0$. By the mtpi, it follows, for $c > 0$, that $c^j \cdot c > 0 \cdot c = 0$. Since, by the r.d., $c^j \cdot c = c^{j+1}$, it follows that, for $c > 0$, if $c^j > 0$ then $c^{j+1} > 0$.

(iii) From (i) and (ii) it follows, for $c > 0$, by Theorem 114, that $\forall_{k \geq 0} c^k \geq 0$.

Consequently, $\forall_{x > 0} \forall_{k \geq 0} x^k > 0$.

In the proof of Theorem 150b, given on pages 8-102 and 8-103, the restrictions necessitated by presence of the restricted quantifier in the recursive definition of the exponential sequences have been omitted. If these were included, the last line on page 8-102 would begin:

(ii) Suppose that [for a $j \geq 0$] $(-1)^{j+2} = (-1)^j$. Since...

Since, on page 8-114, exponentiation is defined for all integral exponents, the need for such restrictions as that just mentioned is temporary. For this reason, such restrictions are omitted, also, in the answers for Part G. For the same reason, Theorems 150 and 152 as given in the theorem list on pages 8-241 and 8-242 have ' \forall_k ' rather than ' $\forall_{k \geq 0}$ '. This more general form of Theorem 150a, for example [cf. Exercise 1 of Part G], is proved in the Sample on page 8-115, and the proofs of the more general forms of the three parts of Theorem 151 are the exercises which follow this sample.

*

Answers for Part G.

1. (i) By the recursive definition for the exponential sequence with base 1, $1^0 = 1$.

(ii) Suppose that $1^j = 1$. Again by the r.d., it follows that $1^{j+1} = 1^j \cdot 1 = 1 \cdot 1 = 1$. Consequently, if $1^j = 1$ then $1^{j+1} = 1$.

(iii) From (i) and (ii) it follows, by Theorem 114, that $\forall_{k \geq 0} 1^k = 1$.

2. (i) By the r.d., $(-1)^{2 \cdot 0} = (-1)^0 = 1$ and $(-1)^{2 \cdot 0 + 1} = (-1)^{0+1} = (-1)^0 \cdot -1 = 1 \cdot -1 = -1$.

(ii) Suppose that $(-1)^{2j} = 1$ and $(-1)^{2j+1} = -1$. By the r.d.,

$$(-1)^{2(j+1)} = (-1)^{(2j+1)+1} = -1 \cdot -1 = 1, \text{ and}$$

$$(-1)^{2(j+1)+1} = (-1)^{2(j+1)} \cdot -1 = 1 \cdot -1 = -1.$$

last line:

$$\begin{aligned}
 & \sum_{k=0}^{p-1} \left(\left[\frac{n}{m^k} \right] - \left[\frac{n}{m^{k+1}} \right] m \right) m^k \\
 &= \sum_{k=0}^{p-1} \left(\left[\frac{n}{m^k} \right] m^k - \left[\frac{n}{m^{k+1}} \right] m^{k+1} \right) \left. \vphantom{\sum_{k=0}^{p-1}} \right\} \text{Theorem 137} \\
 &= \sum_{k=1}^p \left(\left[\frac{n}{m^{k-1}} \right] m^{k-1} - \left[\frac{n}{m^k} \right] m^k \right) \left. \vphantom{\sum_{k=1}^p} \right\} \text{Theorem 138} \\
 &= \left[\frac{n}{m^0} \right] m^0 - \left[\frac{n}{m^p} \right] m^p \\
 &= \left[n \right] - \left[\frac{n}{m^p} \right] m^p \left. \vphantom{\left[\frac{n}{m^p} \right]} \right\} \begin{array}{l} \text{definition of greatest} \\ \text{integer function;} \\ n \in I \end{array} \\
 &= n - \left[\frac{n}{m^p} \right] m^p
 \end{aligned}$$

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argument based on Theorem 126d, that

$$q \mid \sum_{p=2}^q [(q+1)^{p-1} - 1].$$

By another application of Theorem 126d it follows that

$$q \mid \sum_{p=2}^q [(q+1)^{p-1} - 1] + q.$$

Hence, by definition [and the apm],

$$q^2 \mid q \left[\sum_{p=2}^q [(q+1)^{p-1} - 1] + q \right].$$

So, $q^2 \mid (q+1)^q - 1$. Consequently, $\forall_n n^2 \mid (n+1)^n - 1$.

*

page 8-106, lines 8, 9, and 10: If $m = 10$, 2 is a sufficiently large p , $n_0 = 0$ and $n_1 = 5$. If $m = 6$, 3 is a sufficiently large p , $n_0 = 2$, $n_1 = 2$, and $n_2 = 1$ [$50 = 2 \cdot 6^0 + 2 \cdot 6^1 + 1 \cdot 6^2$].

line 5 from bottom: By the theorem displayed on line 7 from the bottom of page 8-106, each of the numbers given by (*) is a difference of integers and, so, by Theorem 110c, is an integer. By the same theorem, each of these numbers is nonnegative [and less than m].

integers then, for each m , $\sum_{p=1}^m a_p \in I^+$. So, since $q+1-1=q$, it

follows, by definition, that $q \mid (q+1)^q - 1$. Consequently,

$\forall_n n \mid (n+1)^n - 1$. [Notice that essentially the same proof shows that $\forall_m \forall_n n \mid (n+1)^m - 1$. This is referred to in the Hint for Exercise ★10.]

$$9. \quad 1 [= 1^2 \cdot 1]; \quad 8 [= 2^2 \cdot 2]; \quad 63 [= 3^2 \cdot 7]; \quad 624 [= 4^2 \cdot 39];$$

$$7775 [= 5^2 \cdot 315]; \quad \forall_n n^2 \mid (n+1)^n - 1$$

★10. By Theorem 153.

$$\begin{aligned}
 (q+1)^q - 1 &= q \sum_{p=1}^q (q+1)^{p-1} \\
 &= q \sum_{p=1}^q [(q+1)^{p-1} - 1] + q \sum_{p=1}^q 1 \\
 &= q \sum_{p=1}^q [(q+1)^{p-1} - 1] + q^2 \\
 &= q \left[\sum_{p=1}^q [(q+1)^{p-1} - 1] + q \right] \\
 &= q \left[\sum_{p=2}^q [(q+1)^{p-1} - 1] + q \right]
 \end{aligned}
 \begin{array}{l}
 \left. \begin{array}{l} \text{Theorem 134} \\ \text{Theorem 131a} \end{array} \right\} \\
 \left. \begin{array}{l} \\ \text{Theorem 136} \end{array} \right\}
 \end{array}$$

Since, for $p \geq 2$, $q \mid (q+1)^{p-1} - 1$, it follows, by an easy inductive

5. Since \overline{OB} is the altitude to the hypotenuse of right triangle ABC, $OB^2 = OA \cdot OC$. So, $b^2 = 1 \cdot OC = OC$. Similarly, $OC^2 = OB \cdot OD$. So, $b^4 = b \cdot OD$ and $OD = b^3$. Also, $OD^2 = OC \cdot OE$. Thus, $b^6 = b^2 \cdot OE$ and $OE = b^4$. [This construction can be continued, thus giving us a way of generating powers geometrically. The diagram is drawn for a case in which $b > 1$. What would the diagram look like if $b = 1$? If $b < 1$? There are other questions one might ask in connection with this problem. For example, draw $\overrightarrow{AD'}$ such that $\angle BAD'$ is a right angle and $D' \in \overrightarrow{OD}$. Then $OD' = 1/b$. Continuing in this fashion, $OC' = 1/b^2$, $OB' = 1/b^3$, etc..] See the COMMENTARY for Exercise E1 on page 8-136.]

6. [Note that a direct application of Theorem 153 would tell you, for example, that a factorization of $x^3 - 1$ is $(x - 1)(1 + x + x^2)$. The form we give is the conventional one.] [It is instructive to obtain the factorizations by using the division-with-remainder algorithm.]

(a) $(x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$

(b) $(y - 1)(y^5 + y^4 + y^3 + y^2 + y + 1)$

(c) $(1 - z)(1 + z + z^2 + z^3)$ [$1 - z^4 = -(z^4 - 1)$]

(d) $(2x - 1)(4x^2 + 2x + 1)$ [$8x^3 - 1 = (2x)^3 - 1$]

7. (a) 2 [= 3 - 1] (b) 3 (c) 4 (d) 9 (e) 99

8. $(n + 1)^n$; By Theorem 153, $(q + 1)^q - 1 = (q + 1 - 1) \sum_{p=1}^q (q + 1)^{p-1}$.

By Theorem 151a, for each p , $(q + 1)^{p-1} \in I^+$ and, by an easy inductive argument based on Theorem 102, if a is a sequence of positive

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DEPARTMENTS

1919

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authorization of the Post Office Department.

$$2. \$28.57 \quad [0.2(x - 0.3x) = (x - 0.3x) - 16 \iff x = 28\frac{4}{7}]$$

$$3. (a) 4$$

$$(b) 6.5$$

$$(c) \frac{6(x-1)}{x(x+3)}$$

$$4. 18, -18$$

[Suppose that the roots are r and $2r$. Then, the equation ' $(x - r)(x - 2r) = 0$ ' is equivalent to the given equation. So, we are looking for a k such that, for each x ,

$$x^2 - 3rx + 2r^2 = x^2 + \frac{k}{2}x + 18,$$

--that is,

$$(\frac{k}{2} + 3r)x + 18 - 2r^2 = 0.$$

Since this last equation holds for each x , it holds for 0. Hence, $18 - 2r^2 = 0$. Taking account of this and of the fact that the last displayed equation holds for 1, it follows that $\frac{k}{2} + 3r = 0$. From these two results we see, first, that r is -3 or 3 and, second, that k is 18 or -18 . So, in order that the conditions of the exercise be satisfied it is necessary that k be 18 or -18 . To see whether this is sufficient for the satisfaction of the given conditions, we substitute, first ' 18 ', then ' -18 ', for ' k ' in the given equation and, in each case, check that the resulting equation has two roots, one of which is twice the other.]

$$5. 12 \text{ minutes}$$

$$6. K = \frac{1}{4}\sqrt{4a^2 - b^2}$$

$$7. x^2 - 4x + 2 = 0 \quad [\iff (x - 2 - \sqrt{2})(x - 2 + \sqrt{2}) = 0]$$

$$8. 59.5 \quad \left[= \frac{7}{10} \cdot \frac{17}{20} \cdot 100 \right]$$

$$9. 24$$

1911

1912

The first of the year was a very dry one, and the crops were much injured by the drought. The weather was very hot, and the crops were much injured by the drought.

The second of the year was a very wet one, and the crops were much injured by the drought. The weather was very hot, and the crops were much injured by the drought.

The third of the year was a very dry one, and the crops were much injured by the drought. The weather was very hot, and the crops were much injured by the drought.

The fourth of the year was a very wet one, and the crops were much injured by the drought. The weather was very hot, and the crops were much injured by the drought.

The fifth of the year was a very dry one, and the crops were much injured by the drought. The weather was very hot, and the crops were much injured by the drought.

The sixth of the year was a very wet one, and the crops were much injured by the drought. The weather was very hot, and the crops were much injured by the drought.

The seventh of the year was a very dry one, and the crops were much injured by the drought. The weather was very hot, and the crops were much injured by the drought.

$$p - 5 \overline{) \frac{1}{p+13}} \\ \underline{p - 5} \\ 18 = a_4$$

$$p - 6 \overline{) \frac{0}{1}} \\ \underline{0} \\ 1 = a_5$$

So,

$$\begin{aligned} \forall_p p^5 + 3p^4 - p^3 + 2p^2 - 3p + 1 \\ = 3 + 72(p-1) + 161(p-1)(p-2) + 94(p-1)(p-2)(p-3) \\ + 18(p-1)(p-2)(p-3)(p-4) + 1(p-1)(p-2)(p-3)(p-4)(p-5). \end{aligned}$$

By Theorem 139,

$$\begin{aligned} \forall_n \sum_{p=1}^n (p^5 + 3p^4 - p^3 + 2p^2 - 3p + 1) \\ = 3n + 72 \frac{n(n-1)}{2} + 161 \frac{n(n-1)(n-2)}{3} + 94 \frac{n(n-1)(n-2)(n-3)}{4} \\ + 18 \frac{n(n-1)(n-2)(n-3)(n-4)}{5} + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{6}. \end{aligned}$$

*

Answers for Miscellaneous Exercises.

Answers for Part A.

$$\left[\begin{array}{ll} \text{Easy:} & 2, 3, 5-14, 16-19, 21, 23, 24, 27, 29-32; \\ \text{Medium:} & 1, 15, 25, 26; \\ \text{Hard:} & 4, 20, 22, 28 \end{array} \right]$$

- Suppose that $AB = 2x$. Then, $EA = x$ and, by the Pythagorean Theorem, $EB = x\sqrt{5}$. So, since $EB = EF$, $AF = x\sqrt{5} - x = x(\sqrt{5} - 1)$, and the area-measure of the square $AFGH$ is $[x(\sqrt{5} - 1)]^2$, or $x^2(6 - 2\sqrt{5})$. On the other hand, since $AH = AF$, $HB = 2x - x(\sqrt{5} - 1)$, or $x(3 - \sqrt{5})$. Also, since $\angle GHA$ is a right angle, quadrilateral $KHBC$ is a rectangle and, so, $KH = 2x$. Hence, the area-measure of quadrilateral $KHBC$ is $2x^2(3 - \sqrt{5})$, or $x^2(6 - 2\sqrt{5})$.

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In this way, any polynomial expression in 'x' can be transformed into a polynomial expression in any binomial of the form 'x - h'.

As another application, consider the problem of proving a summation theorem by the method of difference-sequences. For example:

$$\forall_n \sum_{p=1}^n (p^5 + 3p^4 - p^3 + 2p^2 - 3p + 1) = ?$$

This problem could be solved easily by using Theorem 139, if the polynomial 'p⁵ + 3p⁴ - p³ + 2p² - 3p + 1' were transformed into an expression of the form:

$$a_0 + a_1(p-1) + a_2(p-1)(p-2) + a_3(p-1)(p-2)(p-3) \\ + a_4(p-1)(p-2)(p-3)(p-4) + a_5(p-1)(p-2)(p-3)(p-4)(p-5)$$

This can be done by successive application of the division-with-remainder algorithm:

$$\begin{array}{r} p^4 + 4p^3 + 3p^2 + 5p + 2 \\ p-1 \overline{) p^5 + 3p^4 - p^3 + 2p^2 - 3p + 1} \\ \underline{p^5 - p^4} \\ 4p^4 - p^3 \\ \underline{4p^4 - 4p^3} \\ 3p^3 + 2p^2 \\ \underline{3p^3 - 3p^2} \\ 5p^2 - 3p \\ \underline{5p^2 - 5p} \\ 2p + 1 \\ \underline{2p - 2} \\ 3 = a_0 \end{array}$$

$$\begin{array}{r} p^3 + 6p^2 + 15p + 35 \\ p-2 \overline{) p^4 + 4p^3 + 3p^2 + 5p + 2} \\ \underline{p^4 - 2p^3} \\ 6p^3 + 3p^2 \\ \underline{6p^3 - 12p^2} \\ 15p^2 + 5p \\ \underline{15p^2 - 30p} \\ 35p + 2 \\ \underline{35p - 70} \\ 72 = a_1 \end{array}$$

$$\begin{array}{r} p^2 + 9p + 42 \\ p-3 \overline{) p^3 + 6p^2 + 15p + 35} \\ \underline{p^3 - 3p^2} \\ 9p^2 + 15p \\ \underline{9p^2 - 27p} \\ 42p + 35 \\ \underline{42p - 126} \\ 161 = a_2 \end{array}$$

$$\begin{array}{r} p + 13 \\ p-4 \overline{) p^2 + 9p + 42} \\ \underline{p^2 - 4p} \\ 13p + 42 \\ \underline{13p - 52} \\ 94 = a_3 \end{array}$$

A formally similar problem is that of finding numbers $a_0, a_1, a_2, a_3,$ and a_4 such that, for each x ,

$$2 + 3x + 6x^2 + 5x^3 + 7x^4 = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4.$$

As before, these numbers are found as the remainders after successive divisions-with-remainder by ' $x-1$ ':

$$\begin{array}{r} 7x^3 + 12x^2 + 18x + 21 \\ x-1 \overline{) 7x^4 + 5x^3 + 6x^2 + 3x + 2} \\ \underline{7x^4 - 7x^3} \\ 12x^3 + 6x^2 \\ \underline{12x^3 - 12x^2} \\ 18x^2 + 3x \\ \underline{18x^2 - 18x} \\ 21x + 2 \\ \underline{21x - 21} \\ 23 = a_0 \end{array}$$

$$\begin{array}{r} 7x^2 + 19x + 37 \\ x-1 \overline{) 7x^3 + 12x^2 + 18x + 21} \\ \underline{7x^3 - 7x^2} \\ 19x^2 + 18x \\ \underline{19x^2 - 19x} \\ 37x + 21 \\ \underline{37x - 37} \\ 58 = a_1 \end{array}$$

$$\begin{array}{r} 7x + 26 \\ x-1 \overline{) 7x^2 + 19x + 37} \\ \underline{7x^2 - 7x} \\ 26x + 37 \\ \underline{26x - 26} \\ 63 = a_2 \end{array}$$

$$\begin{array}{r} 7 \\ x-1 \overline{) 7x + 26} \\ \underline{7x - 7} \\ 33 = a_3 \end{array}$$

$$\begin{array}{r} 0 \\ x-1 \overline{) 7} \\ \underline{0} \\ 7 = a_4 \end{array}$$

So,

$$\forall_x \quad 2 + 3x + 6x^2 + 5x^3 + 7x^4 = 23 + 58(x-1) + 63(x-1)^2 + 33(x-1)^3 + 7(x-1)^4.$$

[The algorithm is much simplified by the use of "synthetic division":

| | | | | | |
|---|----|----|---------------------|---------------------|---|
| 7 | 5 | 6 | 3 | 2 | 1 |
| | 7 | 12 | 18 | 21 | |
| 7 | 12 | 18 | 21 | 23 = a ₀ | 1 |
| | 7 | 19 | 37 | 58 = a ₁ | 1 |
| 7 | 19 | 37 | 63 = a ₂ | 33 = a ₃ | 1 |
| | 7 | 26 | 7 | 7 = a ₄ | |
| 7 | 26 | 7 | | | |
| | 7 | | | | |

| | | | | | | |
|----|---------------------|-----------|------------------------|-----------|------------------------|-----------|
| 3. | $2 \overline{) 39}$ | | $2 \overline{) 75621}$ | | $2 \overline{) 40320}$ | |
| | $3 \overline{) 19}$ | $1 = n_0$ | $3 \overline{) 37810}$ | $1 = n_0$ | $3 \overline{) 20160}$ | $0 = n_0$ |
| | $4 \overline{) 6}$ | $1 = n_1$ | $4 \overline{) 12603}$ | $1 = n_1$ | $4 \overline{) 6720}$ | $0 = n_1$ |
| | $5 \overline{) 1}$ | $2 = n_2$ | $5 \overline{) 3150}$ | $3 = n_2$ | $5 \overline{) 1680}$ | $0 = n_2$ |
| | 0 | $1 = n_3$ | $6 \overline{) 630}$ | $0 = n_3$ | $6 \overline{) 336}$ | $0 = n_3$ |
| | | | $7 \overline{) 105}$ | $0 = n_4$ | $7 \overline{) 56}$ | $0 = n_4$ |
| | | | $8 \overline{) 15}$ | $0 = n_5$ | $8 \overline{) 8}$ | $0 = n_5$ |
| | | | $9 \overline{) 1}$ | $7 = n_6$ | $9 \overline{) 1}$ | $0 = n_6$ |
| | | | 0 | $1 = n_7$ | 0 | $1 = n_7$ |

So, $39 = 1 \cdot 1! + 1 \cdot 2! + 2 \cdot 3! + 1 \cdot 4!;$

$75621 = 1 \cdot 1! + 1 \cdot 2! + 3 \cdot 3! + 0 \cdot 4! + 0 \cdot 5! + 0 \cdot 6! + 7 \cdot 7! + 1 \cdot 8!;$

$40320 = 0 \cdot 1! + 0 \cdot 2! + 0 \cdot 3! + 0 \cdot 4! + 0 \cdot 5! + 0 \cdot 6! + 0 \cdot 7! + 1 \cdot 8!.$

*

The generalization discussed in the answer for Exercise 1 of Part B [replacing the exponential sequence with base m by any sequence a of positive integers such that $a_0 = 1$, $\forall_k a_k | a_{k+1}$, and $\forall_n \exists_p a_p > n$] leads to an algorithm for finding the digits in the "a-representation" of an integer. As illustrated in the answers for Exercise 3 of Part B, the digit-numbers n_0, n_1, n_2, \dots are the remainders after succession divisions by $a_1/a_0, a_2/a_1, a_3/a_2, \dots$.

This algorithm has interesting extensions to problems of a more algebraic nature. For example, finding the base-9 representation of the integer whose base-10 representation is '75632' amounts to finding digits, n_0, n_1, n_2, n_3, n_4 , and n_5 such that

$$\begin{aligned}
 & 2 + 3 \cdot 10 + 6 \cdot 10^2 + 5 \cdot 10^3 + 7 \cdot 10^4 \\
 &= n_0 + n_1(10 - 1) + n_2(10 - 1)^2 + n_3(10 - 1)^3 \\
 &\quad + n_4(10 - 1)^4 + n_5(10 - 1)^5.
 \end{aligned}$$

These digits are found as the remainders after successive divisions-with remainder by $10 - 1$.

Since a nonnegative number less than 1 is its own fractional part, it follows that

$$\frac{1}{m} \leq \left\{ \left\lfloor \frac{n/m^{p-1}}{m} \right\rfloor \right\}$$

and, so, that $n_{p-1} \geq \frac{1}{m} \cdot m = 1$. Hence, $n_{p-1} \neq 0$.

*

Answers for Part B.

1. The discussion of base- m representations can be repeated with the exponential sequence with base m replaced by any sequence a of positive integers such that $a_0 = 1$, $\forall_{k \geq 0} a_k | a_{k+1}$, and $\forall_n \exists_p a_p > n$. All that is required is to replace, in the displays on pages 8-106 and 8-107, ' m^k ' by ' a_k ', ' m ' by ' a_{k+1}/a_k ', ' m^{k+1} ' by ' a_{k+1} ', and ' m^p ' by ' a_p '. [The interesting cases are those in which, for each k , $a_{k+1}/a_k \neq 1$. For, if $a_{k+1}/a_k = 1$ then, since

$$n_k = \left\lfloor \frac{n}{a_k} \right\rfloor - \left\lfloor \frac{n}{a_{k+1}} \right\rfloor \frac{a_{k+1}}{a_k},$$

it follows that, for each n , $n_k = 0$.]

The sequence a such that, for each k , $a_k = (k+1)!$ has all four of the properties mentioned above. [It is to obtain the last property that we make use of this "translated factorial sequence", rather than the factorial sequence itself.] That this is the case follows readily from the recursive definition of the factorial sequence and Exercise 4(a), both of which are in Part D on page 8-98. [For the third property, note that by Exercise 4(a), $(n+1)! \geq n+1$ --and that $n+1 > n$.]

2. Since, as noted just above, $(n+1)! > n$, one "sufficiently large p " is n , itself.

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Although the exercises on page 8-107 are marked as optional, students should do Exercise 1 of Part A.

*

Answers for Part A.

1. (i) $p^0 = 1 > 0$

(ii) Suppose that $p^j > j$. Since, by Theorem 152c, for $p > 1$, $p^{j+1} > p^j$ and since by Theorem 151a, $p^{j+1} \in I^+$ and $p^j \in I^+$, it follows from Theorem 106 that, for $p > 1$, $p^{j+1} \geq p^j + 1$. Since, by hypothesis, $p^j > j$, it follows from the atpi that $p^j + 1 > j + 1$. So, by Theorem 86c, $p^{j+1} > j + 1$. Hence, for $p > 1$, if $p^j > j$ then $p^{j+1} > j + 1$.

(iii) From (i) and (ii) it follows, by Theorem 114, that, for $p > 1$,
 $\forall_{k \geq 0} p^k > k$.

Consequently, $\forall_{m > 1} \forall_{k \geq 0} m^k > k$.

2. [It follows (for $m > 1$) from Theorem 151b that $m^n > n$. So, there is at least one integer $k \geq 0$ such that $m^k > n$. Hence, by Theorem 113, there is a least integer $k \geq 0$ such that $m^k > n$. Since $m^0 = 1 \not> n$, this least integer is some positive integer p .] By (*) in page 8-106,

$$n_{p-1} = \left\{ \left\lfloor \frac{n/m^{p-1}}{m} \right\rfloor \right\} m.$$

Since $m^{p-1} \leq n < m^p$, it follows from Theorem 152a and the mtpi that

$$1 \leq n/m^{p-1} < m.$$

Hence,

$$1 \leq \lfloor n/m^{p-1} \rfloor < m,$$

and

$$\frac{1}{m} \leq \frac{\lfloor n/m^{p-1} \rfloor}{m} < 1.$$

$$(i) \frac{2}{3} \left[1 - \frac{1}{4} \right] = \frac{1}{2} = P_1$$

$$\begin{aligned}
 (ii) P_{q+1} &= P_q + \frac{(-1)^{q+1}}{2^{q+1}} \\
 &= \frac{2}{3} \left[1 - \left(-\frac{1}{2} \right)^{q+1} \right] + \left(-\frac{1}{2} \right)^{q+1} \\
 &= \frac{2}{3} + \frac{1}{3} \left(-\frac{1}{2} \right)^{q+1} \\
 &= \frac{2}{3} + \frac{1}{3} \cdot \frac{\left(-\frac{1}{2} \right)^{q+2}}{-\frac{1}{2}} \\
 &= \frac{2}{3} \left[1 - \left(-\frac{1}{2} \right)^{q+2} \right]
 \end{aligned}
 \left. \vphantom{\begin{aligned} P_{q+1} &= P_q + \frac{(-1)^{q+1}}{2^{q+1}} \\ &= \frac{2}{3} \left[1 - \left(-\frac{1}{2} \right)^{q+1} \right] + \left(-\frac{1}{2} \right)^{q+1} \\ &= \frac{2}{3} + \frac{1}{3} \left(-\frac{1}{2} \right)^{q+1} \end{aligned}} \right\} \begin{array}{l} \text{[inductive} \\ \text{hypothesis]} \end{array}$$

So, the distance of the point from its original position after n seconds is

$$|1 - P_n| = \frac{1}{3} + \frac{2}{3} \left(-\frac{1}{2} \right)^{n+1} = \frac{1}{3} \left[1 - \left(-\frac{1}{2} \right)^n \right].$$

For the second part of the problem, we note that the shortest distance moved during the q th second is $|P_q - P_{q-1}|$. So,

$$\begin{aligned}
 \sum_{q=1}^n |P_q - P_{q-1}| &= \sum_{q=1}^n \left(\frac{1}{2} \right)^q = \frac{1}{2} \sum_{q=1}^n \left(\frac{1}{2} \right)^{q-1} \\
 &= \frac{1}{2} \cdot \frac{\left(\frac{1}{2} \right)^n - 1}{\frac{1}{2} - 1} \\
 &= 1 - \left(\frac{1}{2} \right)^n.
 \end{aligned}$$

27. $x^2 - 60x + 576 = 0$

[Students should see the following attack:

Suppose that r_1 and r_2 are the roots of the sought-for quadratic.

Then, it is equivalent to:

$$(x - r_1)(x - r_2) = 0$$

or to:

$$x^2 - (r_1 + r_2)x + r_1r_2 = 0$$

But, $\frac{r_1 + r_2}{2} = 30$ and $\sqrt{r_1r_2} = 24$.

So, ...

This eliminates the need for finding the roots.]

28. The distance of the point from its original position after n seconds is $|1 - P_n|$. As in Exercise 22, we do some computing and search for a pattern in order to find an explicit definition for P .

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-------|---------------|---------------|---------------|-----------------|-----------------|-----------------|------------------|-------------------|
| P_n | $\frac{1}{2}$ | $\frac{3}{4}$ | $\frac{5}{8}$ | $\frac{11}{16}$ | $\frac{21}{32}$ | $\frac{43}{64}$ | $\frac{85}{128}$ | $\frac{171}{256}$ |

After some experimenting, we note that if each numerator-number is tripled, the result differs by 1 from a power of 2. In fact, it appears to be the case that, for each n ,

$$\begin{aligned} P_n &= \frac{1}{3} \cdot \frac{2^{n+1} - (-1)^{n+1}}{2^n} \\ &= \frac{2}{3} \cdot \frac{2^{n+1} - (-1)^{n+1}}{2^{n+1}} \\ &= \frac{2}{3} \left[1 - \left(-\frac{1}{2} \right)^{n+1} \right]. \end{aligned}$$

[This last expression makes it pretty clear that the point is oscillating about $\frac{2}{3}$ and keeps getting closer to $\frac{2}{3}$ as n increases.]

The derivation of this formula for p_n from the recursive definition is easily carried out by mathematical induction:

Let $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$ be a family of subsets of S . Then \mathcal{A} is called a λ -system if

$$A_i \cap A_j \in \mathcal{A} \quad \text{for all } i, j \in \{1, 2, \dots, n\}$$

$$\text{and } A_i \cup A_j \in \mathcal{A} \quad \text{for all } i, j \in \{1, 2, \dots, n\}$$

$$\text{if } A_i \in \mathcal{A} \text{ then } A_i^c \in \mathcal{A} \quad \text{for all } i \in \{1, 2, \dots, n\}$$

Let \mathcal{A} be a λ -system. Then \mathcal{A} is called a σ -system if

$$A_i \in \mathcal{A} \text{ implies } A_i^c \in \mathcal{A} \quad \text{for all } i \in \{1, 2, \dots, n\}$$

Let \mathcal{A} be a σ -system. Then \mathcal{A} is called a σ -algebra if

$$A_i \in \mathcal{A} \text{ implies } A_i^c \in \mathcal{A} \quad \text{for all } i \in \{1, 2, \dots, n\}$$

$$\text{Let } \mathcal{A} \text{ be a } \sigma\text{-algebra. Then } \mathcal{A} \text{ is called a } \sigma\text{-algebra if } A_i \in \mathcal{A} \text{ implies } A_i^c \in \mathcal{A} \text{ for all } i \in \{1, 2, \dots, n\}$$

$$\text{Let } \mathcal{A} \text{ be a } \sigma\text{-algebra. Then } \mathcal{A} \text{ is called a } \sigma\text{-algebra if } A_i \in \mathcal{A} \text{ implies } A_i^c \in \mathcal{A} \text{ for all } i \in \{1, 2, \dots, n\}$$

$$\text{Let } \mathcal{A} \text{ be a } \sigma\text{-algebra. Then } \mathcal{A} \text{ is called a } \sigma\text{-algebra if } A_i \in \mathcal{A} \text{ implies } A_i^c \in \mathcal{A} \text{ for all } i \in \{1, 2, \dots, n\}$$

$$\text{Let } \mathcal{A} \text{ be a } \sigma\text{-algebra. Then } \mathcal{A} \text{ is called a } \sigma\text{-algebra if } A_i \in \mathcal{A} \text{ implies } A_i^c \in \mathcal{A} \text{ for all } i \in \{1, 2, \dots, n\}$$

$$\text{Let } \mathcal{A} \text{ be a } \sigma\text{-algebra. Then } \mathcal{A} \text{ is called a } \sigma\text{-algebra if } A_i \in \mathcal{A} \text{ implies } A_i^c \in \mathcal{A} \text{ for all } i \in \{1, 2, \dots, n\}$$

$$\text{Let } \mathcal{A} \text{ be a } \sigma\text{-algebra. Then } \mathcal{A} \text{ is called a } \sigma\text{-algebra if } A_i \in \mathcal{A} \text{ implies } A_i^c \in \mathcal{A} \text{ for all } i \in \{1, 2, \dots, n\}$$

$$\text{Let } \mathcal{A} \text{ be a } \sigma\text{-algebra. Then } \mathcal{A} \text{ is called a } \sigma\text{-algebra if } A_i \in \mathcal{A} \text{ implies } A_i^c \in \mathcal{A} \text{ for all } i \in \{1, 2, \dots, n\}$$

$$\text{Let } \mathcal{A} \text{ be a } \sigma\text{-algebra. Then } \mathcal{A} \text{ is called a } \sigma\text{-algebra if } A_i \in \mathcal{A} \text{ implies } A_i^c \in \mathcal{A} \text{ for all } i \in \{1, 2, \dots, n\}$$

$$\text{Let } \mathcal{A} \text{ be a } \sigma\text{-algebra. Then } \mathcal{A} \text{ is called a } \sigma\text{-algebra if } A_i \in \mathcal{A} \text{ implies } A_i^c \in \mathcal{A} \text{ for all } i \in \{1, 2, \dots, n\}$$

25. 70 [Since $f_n - f_{n-1} = 22 - 3n$, it follows that $f_n > f_{n-1}$ if and only if $n < \frac{22}{3}$ -- that is, if and only if $n \leq 7$. So, the largest value of f is f_7 . $f_7 = 22 \cdot 7 - 3 \cdot \frac{7 \cdot 8}{2} = (22 - 12)7 = 70$. Since 70 is the largest value of f , it is the least upper bound of the range of f .

A second procedure is to note that, for each n , $f_n = \frac{41n - 3n^2}{2}$, and that the quadratic function defined by ' $41x - 3x^2$ ' has its maximum at $41/6$. From the symmetry of the function it follows, again, that the greatest value of f is f_7 .]

$$26. (i) \quad \sum_{p=1+1}^{2 \cdot 1} \frac{1}{p} = \frac{1}{2}; \quad \sum_{p=1}^{2 \cdot 1} \frac{(-1)^{p-1}}{p} = \frac{(-1)^{1-1}}{1} + \frac{(-1)^{2-1}}{2} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\begin{aligned}
 (ii) \quad \sum_{p=q+2}^{2q+2} \frac{1}{p} &= \sum_{p=q+1}^{2q+2} \frac{1}{p} - \frac{1}{q+1} \\
 &= \sum_{p=q+1}^{2q} \frac{1}{p} + \frac{1}{2q+1} + \frac{1}{2q+2} - \frac{1}{q+1} \quad \left. \vphantom{\sum_{p=q+1}^{2q} \frac{1}{p}} \right\} \text{inductive hypothesis} \\
 &= \sum_{p=1}^{2q} \frac{(-1)^{p-1}}{p} + \frac{1}{2q+1} + \frac{1}{2q+2} - \frac{1}{q+1} \\
 &= \sum_{p=1}^{2q} \frac{(-1)^{p-1}}{p} + \frac{1}{2q+1} + \frac{-1}{2q+2} \quad \left. \vphantom{\sum_{p=1}^{2q} \frac{(-1)^{p-1}}{p}} \right\} \text{Theorem 150c} \\
 &= \sum_{p=1}^{2q} \frac{(-1)^{p-1}}{p} + \frac{(-1)^{2q+1-1}}{2q+1} + \frac{(-1)^{2q+2-1}}{2q+2} \\
 &= \sum_{p=1}^{2q+2} \frac{(-1)^{p-1}}{p}
 \end{aligned}$$

The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \frac{1}{2} (f(x-1) + f(x+1))$. It is shown that $f(x)$ is a linear function of x and that the constant term in this linear function is equal to $f(0)$.

$$f(x) = \frac{1}{2} (f(x-1) + f(x+1))$$

where

is the function defined by the equation

The second part of the paper is devoted to the study of the properties of the function $g(x)$ defined by the equation $g(x) = \frac{1}{2} (g(x-1) + g(x+1))$. It is shown that $g(x)$ is a linear function of x and that the constant term in this linear function is equal to $g(0)$.

$$g(x) = \frac{1}{2} (g(x-1) + g(x+1))$$

where $g(x)$ is the function defined by the equation

$$g(x) = \frac{1}{2} (g(x-1) + g(x+1))$$

The third part of the paper is devoted to the study of the properties of the function $h(x)$ defined by the equation $h(x) = \frac{1}{2} (h(x-1) + h(x+1))$. It is shown that $h(x)$ is a linear function of x and that the constant term in this linear function is equal to $h(0)$.

$$h(x) = \frac{1}{2} (h(x-1) + h(x+1))$$

where

is the function defined by the equation

$$h(x) = \frac{1}{2} (h(x-1) + h(x+1))$$

The fourth part of the paper is devoted to the study of the properties of the function $i(x)$ defined by the equation $i(x) = \frac{1}{2} (i(x-1) + i(x+1))$. It is shown that $i(x)$ is a linear function of x and that the constant term in this linear function is equal to $i(0)$.

$$i(x) = \frac{1}{2} (i(x-1) + i(x+1))$$

$$i(x) = \frac{1}{2} (i(x-1) + i(x+1))$$

$$i(x) = \frac{1}{2} (i(x-1) + i(x+1))$$

where

distance the point can have moved during the q th second is the distance between P_q and P_{q+1} --that is, $|P_{q+1} - P_q|$. From the statement of the problem, this is $q + 1$. So, the shortest total distance it can have moved during the first n seconds is

$$\sum_{q=1}^n (q + 1),$$

that is, $n(n + 3)/2$.

Some students may notice that the intervals $\overline{P_2P_3}$, $\overline{P_4P_5}$, etc. have the same midpoint, $\frac{1}{2}$. And, from this, they may arrive at the result:

$$\sum_n P_n = \frac{1}{2} + (-1)^n \left[\frac{1}{2} + \left\lfloor \frac{n}{2} \right\rfloor \right]$$

Or, they may discover that

$$\sum_n P_n = \frac{3}{4} + (-1)^n \frac{2n + 1}{4}$$

The first result is easy to prove by induction. An inductive proof of the second requires one to show [part (ii)] that

$$\begin{aligned} \frac{1}{2} + (-1)^n \left[\frac{1}{2} + \left\lfloor \frac{n}{2} \right\rfloor \right] + (-1)^{n+1} (n + 1) \\ = \frac{1}{2} + (-1)^{n+1} \left[\frac{1}{2} + \left\lfloor \frac{n+1}{2} \right\rfloor \right] \end{aligned}$$

--that is, that

$$n = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Now, if n is even, $\left\lfloor n/2 \right\rfloor = n/2 = \left\lfloor (n+1)/2 \right\rfloor$, while if n is odd, $\left\lfloor n/2 \right\rfloor = (n-1)/2$ and $\left\lfloor (n+1)/2 \right\rfloor = (n+1)/2$. In either case, the sum is n .

23. 35 miles $\left[\frac{x}{5} - 2 = \frac{15}{5} + 2 \right]$

24. (a) $\frac{2bt + yt + 2dr + ar}{rt(2c + 3)}$

(b) $\frac{x + 10y}{5(5x + y)(x + 5y)(x + y)}$

1. *Phragmites* (Common Reed)

1911

Oxyechinus

1871

1971.01.20

1. *Journal of the American Medical Association*, 1911, 56, 1000.

1991

$$f_j = \sqrt{2\pi} \exp\left(-\frac{1}{2} \left(\frac{1}{\sigma_j^2} \left(\frac{1}{\sigma_j^2} - \frac{1}{\sigma_j^2} \right)\right)\right)$$
$$d^2\sigma/d\tau^2 = -\sigma^2/\tau^2, \quad (7)$$

9 Dec.

2014

Chrysomelidae

1. *Phragmites australis* (Cav.) Trin. ex Steud.

1. 2. 3. 4. 5. 6. 7. 8. 9. 10. 11. 12. 13. 14. 15. 16. 17. 18. 19. 20. 21. 22. 23. 24. 25. 26. 27. 28. 29. 30. 31. 32. 33. 34. 35. 36. 37. 38. 39. 40. 41. 42. 43. 44. 45. 46. 47. 48. 49. 50. 51. 52. 53. 54. 55. 56. 57. 58. 59. 60. 61. 62. 63. 64. 65. 66. 67. 68. 69. 70. 71. 72. 73. 74. 75. 76. 77. 78. 79. 80. 81. 82. 83. 84. 85. 86. 87. 88. 89. 90. 91. 92. 93. 94. 95. 96. 97. 98. 99. 100. 101. 102. 103. 104. 105. 106. 107. 108. 109. 110. 111. 112. 113. 114. 115. 116. 117. 118. 119. 120. 121. 122. 123. 124. 125. 126. 127. 128. 129. 130. 131. 132. 133. 134. 135. 136. 137. 138. 139. 140. 141. 142. 143. 144. 145. 146. 147. 148. 149. 150. 151. 152. 153. 154. 155. 156. 157. 158. 159. 160. 161. 162. 163. 164. 165. 166. 167. 168. 169. 170. 171. 172. 173. 174. 175. 176. 177. 178. 179. 180. 181. 182. 183. 184. 185. 186. 187. 188. 189. 190. 191. 192. 193. 194. 195. 196. 197. 198. 199. 200. 201. 202. 203. 204. 205. 206. 207. 208. 209. 210. 211. 212. 213. 214. 215. 216. 217. 218. 219. 220. 221. 222. 223. 224. 225. 226. 227. 228. 229. 230. 231. 232. 233. 234. 235. 236. 237. 238. 239. 240. 241. 242. 243. 244. 245. 246. 247. 248. 249. 250. 251. 252. 253. 254. 255. 256. 257. 258. 259. 260. 261. 262. 263. 264. 265. 266. 267. 268. 269. 270. 271. 272. 273. 274. 275. 276. 277. 278. 279. 280. 281. 282. 283. 284. 285. 286. 287. 288. 289. 290. 291. 292. 293. 294. 295. 296. 297. 298. 299. 300. 301. 302. 303. 304. 305. 306. 307. 308. 309. 310. 311. 312. 313. 314. 315. 316. 317. 318. 319. 320. 321. 322. 323. 324. 325. 326. 327. 328. 329. 330. 331. 332. 333. 334. 335. 336. 337. 338. 339. 340. 341. 342. 343. 344. 345. 346. 347. 348. 349. 350. 351. 352. 353. 354. 355. 356. 357. 358. 359. 360. 361. 362. 363. 364. 365. 366. 367. 368. 369. 370. 371. 372. 373. 374. 375. 376. 377. 378. 379. 380. 381. 382. 383. 384. 385. 386. 387. 388. 389. 390. 391. 392. 393. 394. 395. 396. 397. 398. 399. 400. 401. 402. 403. 404. 405. 406. 407. 408. 409. 410. 411. 412. 413. 414. 415. 416. 417. 418. 419. 420. 421. 422. 423. 424. 425. 426. 427. 428. 429. 430. 431. 432. 433. 434. 435. 436. 437. 438. 439. 440. 441. 442. 443. 444. 445. 446. 447. 448. 449. 450. 451. 452. 453. 454. 455. 456. 457. 458. 459. 460. 461. 462. 463. 464. 465. 466. 467. 468. 469. 470. 471. 472. 473. 474. 475. 476. 477. 478. 479. 480. 481. 482. 483. 484. 485. 486. 487. 488. 489. 490. 491. 492. 493. 494. 495. 496. 497. 498. 499. 500. 501. 502. 503. 504. 505. 506. 507. 508. 509. 510. 511. 512. 513. 514. 515. 516. 517. 518. 519. 520. 521. 522. 523. 524. 525. 526. 527. 528. 529. 530. 531. 532. 533. 534. 535. 536. 537. 538. 539. 540. 541. 542. 543. 544. 545. 546. 547. 548. 549. 550. 551. 552. 553. 554. 555. 556. 557. 558. 559. 560. 561. 562. 563. 564. 565. 566. 567. 568. 569. 570. 571. 572. 573. 574. 575. 576. 577. 578. 579. 580. 581. 582. 583. 584. 585. 586. 587. 588. 589. 590. 591. 592. 593. 594. 595. 596. 597. 598. 599. 600. 601. 602. 603. 604. 605. 606. 607. 608. 609. 610. 611. 612. 613. 614. 615. 616. 617. 618. 619. 620. 621. 622. 623. 624. 625. 626. 627. 628. 629. 630. 631. 632. 633. 634. 635. 636. 637. 638. 639. 640. 641. 642. 643. 644. 645. 646. 647. 648. 649. 650. 651. 652. 653. 654. 655. 656. 657. 658. 659. 660. 661. 662. 663. 664. 665. 666. 667. 668. 669. 670. 671. 672. 673. 674. 675. 676. 677. 678. 679. 680. 681. 682. 683. 684. 685. 686. 687. 688. 689. 690. 691. 692. 693. 694. 695. 696. 697. 698. 699. 700. 701. 702. 703. 704. 705. 706. 707. 708. 709. 710. 711. 712. 713. 714. 715. 716. 717. 718. 719. 720. 721. 722. 723. 724. 725. 726. 727. 728. 729. 730. 731. 732. 733. 734. 735. 736. 737. 738. 739. 740. 741. 742. 743. 744. 745. 746. 747. 748. 749. 750. 751. 752. 753. 754. 755. 756. 757. 758. 759. 760. 761. 762. 763. 764. 765. 766. 767. 768. 769. 770. 771. 772. 773. 774. 775. 776. 777. 778. 779. 780. 781. 782. 783. 784. 785. 786. 787. 788. 789. 790. 791. 792. 793. 794. 795. 796. 797. 798. 799. 800. 801. 802. 803. 804. 805. 806. 807. 808. 809. 810. 811. 812. 813. 814. 815. 816. 817. 818. 819. 820. 821. 822. 823. 824. 825. 826. 827. 828. 829. 830. 831. 832. 833. 834. 835. 836. 837. 838. 839. 840. 84

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22. Let d_n be the distance of the point from its original position [0] after n seconds. Then, $d_n = |P_{n+1}|$. We need a formula for d . Construct a table, and look for a pattern. [A sketch of the number line may also help.]

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|----|---|----|---|----|---|----|----|
| P_n | 0 | 2 | -1 | 3 | -2 | 4 | -3 | 5 | -4 | 6 |
| d_n | 2 | 1 | 3 | 2 | 4 | 3 | 5 | 4 | 6 | 5 |

Clearly, in looking for a formula for d , it will be helpful to distinguish between the case in which n is odd and that in which n is even. Recalling that the p th positive odd number is $2p - 1$ and that the p th positive even number is $2p$, it is easy to see that, for each p , $d_{2p-1} = p + 1$ and $d_{2p} = p$.

So,
if n is odd then $d_n = \frac{n+1}{2} + 1$ and if n is even then $d_n = \frac{n}{2}$.

[To derive this result from the statement of the exercise, one can begin by proving:

$$(*) \quad \forall_p (P_{2p} = p + 1 \text{ and } P_{2p+1} = -p)$$

This follows by induction. For, (i), from the table, $P_{2 \cdot 1} = 2 = 1 + 1$ and $P_{2 \cdot 1 + 1} = -1$, and, (ii), if $P_{2q} = q + 1$ and $P_{2q+1} = -q$ then

$$P_{2(q+1)} = P_{2q+1} + 2(q+1) = -q + 2(q+1) = q + 2, \text{ and}$$

$$P_{2(q+1)+1} = P_{2(q+1)} - [2(q+1) + 1] = q + 2 - [2(q+1) + 1] = -(q+1).$$

Having established (*), it follows, since $d_n = |P_{n+1}|$, that

$$\forall_p (d_{2p-1} = p + 1 \text{ and } d_{2p} = p).$$

Hence, if $n = 2p - 1$ then $d_n = p + 1 = \frac{n+1}{2} + 1$ and if $n = 2p$ then $d_n = p = \frac{n}{2}$.]

The second part of the problem is somewhat easier. The least

10. The first part of the proof is devoted to the construction of a sequence of functions f_n such that

$$f_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right) \quad (10.1)$$

$$f_n(x) \rightarrow f(x) \quad (10.2)$$

and

$$f_n(x) \rightarrow f(x) \quad (10.3)$$

$$f_n(x) \rightarrow f(x) \quad (10.4)$$

$$f_n(x) \rightarrow f(x) \quad (10.5)$$

$$f_n(x) \rightarrow f(x) \quad (10.6)$$

$$f_n(x) \rightarrow f(x) \quad (10.7)$$

$$f_n(x) \rightarrow f(x) \quad (10.8)$$

$$f_n(x) \rightarrow f(x) \quad (10.9)$$

$$f_n(x) \rightarrow f(x) \quad (10.10)$$

$$f_n(x) \rightarrow f(x) \quad (10.11)$$

$$f_n(x) \rightarrow f(x) \quad (10.12)$$

$$f_n(x) \rightarrow f(x) \quad (10.13)$$

$$f_n(x) \rightarrow f(x) \quad (10.14)$$

$$f_n(x) \rightarrow f(x) \quad (10.15)$$

$$f_n(x) \rightarrow f(x) \quad (10.16)$$

$$f_n(x) \rightarrow f(x) \quad (10.17)$$

$$f_n(x) \rightarrow f(x) \quad (10.18)$$

$$f_n(x) \rightarrow f(x) \quad (10.19)$$

$$f_n(x) \rightarrow f(x) \quad (10.20)$$

$$f_n(x) \rightarrow f(x) \quad (10.21)$$

$$10. A = 12 - B \quad 11. \text{ no roots} \quad 12. (3500 + 250t) \text{ dollars per year}$$

$$13. (a) \frac{(x+4)(x-3)}{(x+5)(x-2)} \quad (b) \frac{(b+c)(b-c)}{(a+c)(a-c)}$$

$$14. \frac{n(n+1)(4n-1)}{6} \left[= 2n \sum_{p=1}^n p - \sum_{p=1}^n p^2 \right]$$

$$15. \left. \begin{aligned} -\frac{1}{3} &< \frac{2c+2}{c^2+3} < 1 \\ \Leftrightarrow -c^2-3 &< 6c+6 < 3c^2+9 \\ \Leftrightarrow -c^2-3 &< 6c+6 \text{ and } 6c+6 < 3c^2+9 \\ \Leftrightarrow (c+3)^2 &> 0 \quad \text{and } (c-1)^2 > 0 \\ \Leftrightarrow c \neq -3 &\quad \text{and } c \neq 1 \\ \Leftrightarrow -3 \neq c \neq 1 \end{aligned} \right\} [3(c^2+3) > 0]$$

$$16. 2y - x \text{ [By Theorem 141b, } \frac{K_{300} - K_{200}}{300 - 200} = \frac{K_{200} - K_{100}}{200 - 100} \cdot \text{]}$$

$$17. (a) 31/9 \quad (b) 30/49 \quad 18. (a) -q/p \quad (b) (a+2)/[2a(a-2)]$$

$$19. 11 \quad [0 = \frac{n}{2}[2 \cdot 10 - 2(n-1)] \Leftrightarrow n(11-n) = 0]$$

20. Weigh 3 coins against 3. Either (a) they balance, or (b) one group of 3 is lighter than the other. In case (a), balance remaining 2 coins, one against the other. Either (a₁) they balance, and there is no underweight coin, or (a₂) they don't, and the lighter coin is underweight. In case (b), balance 2 of the lighter group of 3 coins, one against the other. Either (b₁) they balance, and the third coin is underweight or (b₂) they don't, and the lighter coin is underweight.

$$21. (a) 5x^6 + 3x^5 + 37x^4 + 45x^3 + 57x^2 + 47x + 10 \\ (b) 30x^5 + 15x^4 + 67x^3 + 27x^2 + 38x + 5$$

$$f(x) = g(x) + h(x)$$

$$\begin{aligned} \frac{d}{dx} f(x) &= \frac{d}{dx} (g(x) + h(x)) \\ &= \frac{d}{dx} g(x) + \frac{d}{dx} h(x) \\ &= g'(x) + h'(x) \end{aligned}$$

$$f(x) = g(x) + h(x) \quad \Rightarrow \quad f'(x) = g'(x) + h'(x)$$

$$\begin{aligned} f(x) &= g(x) + h(x) \\ f'(x) &= g'(x) + h'(x) \end{aligned}$$

$$f(x) = g(x) + h(x) \quad \Rightarrow \quad f'(x) = g'(x) + h'(x)$$

$$f(x) = g(x) + h(x) \quad \Rightarrow \quad f'(x) = g'(x) + h'(x)$$

$$(x^2)^4 \quad (x^4)^2 \quad x(xx)^3x \quad (x^2x^2)^2$$

$$\begin{array}{llll} (y^4x)^2x & x^3(y^2)^4 & xxxxyyyy^5 & x^9y^{12} \div (x^3y^2)^2 \\ (y^2)^3x^3y^2 & (xy^2)^3y^0y^2 & (xy)^0(xy)^2(xy)^3y^3 \div x^2 & \\ (xy)^3y^5 & x(xy)^2(y^3)^2 & x^2y^4xy^4 & \end{array}$$

$$x^3y^6 \quad x^3y^5y \quad x^4y^7 \div (xy) \quad (x^1y^2)^3 \quad (xy^2)^3 \quad (xy)^3y^3$$

$$\begin{array}{llllll} x^0x^6 & x^5x & x^6 & x^9 \div x^3 & (x^2)^3 & x^4x^2 & (x^3)^2 \\ x^5x^1 \div x^0 & (x^3)^0(x^0)x^6 & x^{12} \div x^6 & x^3x^3 & x^{12} \div (x^0x^1x^2x^3) & & \end{array}$$

$$x^2y^5$$

$$(x^5)^2$$

$$x^3y^3y^2$$

$$x^3x^8$$

$$(x^3y^2)^4 \quad (x^6y^4)^2$$

$$(x^3)^4y^8$$

$$x^5(x^2)^4$$

$$x^2x^3$$

*

Answers for Part C.

| | | | | |
|---|------------------|-------------------------|----------------|----------------|
| 5^3 | $50^3 \div 10^3$ | 5 cubed | $5^8 \div 5^5$ | 5×5^2 |
| the cube of 5 | third power of 5 | $(5 \times 5) \times 5$ | 125 | |
| third power of the second power of the square root of 5 | | | | |

| | | |
|-------------------|---------------------------|------------------|
| fourth power of 5 | 5×5^3 | $5^1 \times 5^3$ |
| $(5^2)^2$ | $(5^2 \times 5^3) \div 5$ | 5^4 |
| | $5^8 \div 5^4$ | |

| | |
|-----------|-------|
| $(4^2)^3$ | 4^6 |
|-----------|-------|

| | | | | | |
|-----------------------|-------|-------------------------|------------------|--------------|-----------|
| $4^3 \times 4$ | 4^4 | $4 \times 4^2 \times 4$ | $4^2 \times 4^2$ | $4^5 \div 4$ | $(2^4)^2$ |
| 4 to the fourth power | 2^8 | fourth power of 4 | | | |

| | | | | |
|-----------|-------|-------|----------------|-------------------------|
| $(2^2)^3$ | 8^2 | 4^3 | $4^8 \div 4^5$ | $2 \times 2 \times 2^4$ |
|-----------|-------|-------|----------------|-------------------------|

| | |
|-----------|-----------|
| $(3^2)^3$ | $(3^3)^2$ |
|-----------|-----------|

| | | | |
|------------------|-------|------------------------------|----------------------|
| fifth power of 3 | 3^5 | $3^2[(3 \times 3) \times 3]$ | 3 to the fifth power |
|------------------|-------|------------------------------|----------------------|

| | |
|---|------------------|
| $4 \times 4 \times 4 \times 4 \times 4$ | fifth power of 4 |
|---|------------------|

| |
|------------------|
| $5^3 \times 5^2$ |
|------------------|

| |
|-------|
| 5^2 |
|-------|

| |
|----------------|
| $8^3 \div 4^3$ |
|----------------|

*

Answers for Part D.

| | | | | |
|-----------------------|------------|------------------|----------------|---------------|
| $(xxx)^3$ | x^9 | $x^6 \cdot x^3$ | $x^9 \div x^0$ | $x \cdot x^8$ |
| $x^1 x^3 x^0 x^2 x^3$ | $x(x^2)^4$ | $(x^2)^3(x^1)^3$ | $x^2 x^7$ | $x^2 x^3 x^4$ |

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W. H. WELLS, JR., M.D.

29. $12/17$ gallons $[0.15(6 - x) + x = 0.25(6)]$

30. (a) 2 (b) 3

31. (a) $n = \frac{q(c - a) - b}{p(c - a)}$ (b) $n = \frac{b(a - c)p - 1}{(a - c)p}$ (c) $n = \frac{a^2p - q^2r^2}{pb^2}$

32. $6 [kr^3 = k(3^3 + 4^3 + 5^3)]$

*

Answers for Part B.

- | | | | |
|--|--|---------------------|---------------------------|
| 1. 2^4 | 2. 2^5 | 3. 3^5 | 4. $1/5^4$ |
| 5. x^5 | 6. x^2 | 7. x^6 | 8. 1 |
| 9. 4^{55} [or: 2^{105}] | 10. $4^{55}3$ [or: $2^{105}3$] | | |
| 11. x^5y | 12. x^5y^3 | 13. -6^{13} | 14. 3^{13} |
| 15. $1/d^5$ | 16. 19^{15} | 17. y^{12} | 18. 2^93^9 [or: 6^9] |
| 19. x^5/z^2 | 20. 3^{34} | 21. 3^{51} | 22. $9^6/5^7$ |
| 23. 9^4r^{11} | 24. $3^{31}3$ | 25. $x^{860}y^{65}$ | |
| 26. $5^26^3100^5$ [or: $2^{13}3^35^{12}$] | 27. $100^7a^2b^6$ [or: $2^{14}5^{14}a^2b^6$] | | |
| 28. 0 | 29. $-(5a)^5$ | 30. $3^5x^7y^5$ | 31. 0 |
| 32. $1/(xyz^2)$ | 33. $a^4b^4c^4$ [or: $(abc)^4$] | 34. $(2.9t)^3$ | |
| 35. $(4.6p)^5$ | 36. $2^63^24^4a^6c^{11}$ [or: $2^{14}3^2a^6c^{11}$] | 37. $(6.7mn)^7$ | |

*

Note that in Exercise 22 of Part B, in accordance with our convention, ' $5^29^5 \div 5^9$ ' is an abbreviation for ' $[(5^29^5) \div 5^9]9$ '. Compare with Exercise 19.

So, Theorem 154 tells you (with the help of Theorem 17) that

$$5^6 = \frac{1}{5^{-6}}$$

just as much as it tells you that

$$5^{-6} = \frac{1}{5^6}$$

*

In working the Examples at the bottom of page 8-114 as well as the exercises in Part A on page 8-115, we trust that students will discover the short cuts which are justified by Theorems 155-160. In fact you may want to augment Part A in order to promote the discoveries, and to ask students to state the generalizations they have discovered. For example, more exercises like 1, 3, 4, 5, 6, 10, 11, and 12 will lead to Theorem 155. More like 2, 7, 8, and 9 will lead to Theorem 156. Transforming expressions like:

$$(x^{-2})^3, (y^5)^{-3}, (z^{-2})^{-3}, (5^{-2})^4(5^{-3})^{-6}$$

will lead to Theorem 157. Transforming expressions like:

$$(xy)^{-5}(xyz)^4, (x^3y^2)^{-4}(x^{-2}y^4)^5, \left(\frac{x}{y}\right)^{-3}\left(\frac{x^2}{y^{-3}}\right)^4$$

will lead to Theorems 158, 159, and 160.

*

Note the 0-exceptions mentioned in connection with Examples 2 and 3 on page 8-114. We suggest you require explicit statements on these in doing Part A, but forget about them in later drill exercises. Note how the text handles this matter in the sentence at the top of page 8-120 [and in the instructions for Part D on page 8-112].

$$(1') \quad 5^{-2} = \frac{1}{5^2}$$

So, (*) is equivalent to an instance of (1) [on page 8-114]. Now, taking reciprocals of both sides of (*), we get:

$$\frac{1}{5^{-2}} = \frac{1}{\frac{1}{5^{-2}}}$$

and, from this:

$$\frac{1}{5^{-2}} = 5^{-2}$$

Hence, since $5^{-2} = 5^2$,

$$(2') \quad 5^2 = \frac{1}{5^{-2}}$$

So, (*) is equivalent to an instance of (2).

Statements (1) and (2) together give us:

$$(**) \quad \forall x \neq 0 \quad \forall k \neq 0 \quad x^{-k} = \frac{1}{x^k}$$

But, as we learn from (3), (**) doesn't say as much as it could. Thus, we remove the restriction on 'k' and get Theorem 154.

*

In interpreting Theorem 154, students should not make the common error of thinking that the entire message of this theorem is that "raising a number to a negative power is the same as raising the number to the corresponding positive power and taking the reciprocal". [This is what the supplement to the recursive definition tells you.] This is the same kind of error as claiming that the complete message of the theorem:

$$\forall x \quad \forall y \quad (-x)(-y) = xy$$

is that "the product of two negative numbers is a positive number". To avoid this misinterpretation of Theorem 154, ask for instances, and be sure to get some like:

$$5^{-3} = \frac{1}{5^3}, \quad 7^{-2} = \frac{1}{7^2}, \quad (-8)^{-0} = \frac{1}{(-8)^0}$$

The objective here is to define powers with negative integral exponents --that is, to extend the domain of the exponential sequences to include the negative integers--in such a way that [at least] the addition formula for exponents will still hold. It turns out that this can be done in just one way, for powers with nonzero bases, and that, when it is done, not only does the addition formula for exponents continue to hold [Theorem 155], but also the multiplication formula and the distributive formula continue to hold [Theorems 157 and 158].

The text on page 8-113 is concerned with discovering a definition which may have the desirable consequences outlined in the preceding paragraph. The discussion culminates in the boxed supplement, on page 8-114, to the definition of the exponential sequences. That, with this definition of integral exponentiation, the addition formula does still hold is proved on pages 8-117 and 8-118.

Be sure that students understand, on completing page 8-114, that what has been shown is that the chosen definition of integral exponentiation is the only one for which there is a chance of the addition formula holding. From then on, until the completion of the proof of Theorem 155, we are [in a sense] in suspense--will it work or won't it?

In the discussion on page 8-113, which motivates the definition on page 8-114, it is important to distinguish between 'negative' and 'opposite'. So, we make use of the superscript '-' for 'negative'. Be careful to read '+' as 'positive', '-' as 'negative', and '-' as 'the opposite of'.

Theorem 154 follows from (1), (2), and (3) on page 8-114 because, by Theorem 86a, $k > 0$ or $k < 0$ or $k = 0$.

An explanation for statement (3) is that since [by Theorem 80] $-0 = 0$, it follows that $a^{-0} = a^0$, and since [by the recursive definition for exponential sequences] $a^0 = 1$, it follows [since $1 \neq 0$], by Theorem 51, that $a^{-0} = \frac{1}{a^0}$.

Some additional light can be cast upon the derivation of Theorem 154 by considering an instance of the supplement [first boxed statement] to the recursive definition. Take the instance:

$$(**) \quad 5^{-2} = \frac{1}{5^{- -2}}$$

Since $- -2 = 2$, (*) is equivalent to:

1. The first part of the report is a general introduction to the subject.

2. The second part is a detailed description of the methods used.

3. The third part is a discussion of the results obtained.

4. The fourth part is a conclusion and a summary of the findings.

5. The fifth part is a list of references.

6. The sixth part is a list of figures and tables.

7. The seventh part is a list of appendices.

8. The eighth part is a list of footnotes.

9. The ninth part is a list of symbols and abbreviations.

10. The tenth part is a list of acknowledgments.

11. The eleventh part is a list of the author's address and contact information.

12. The twelfth part is a list of the author's other works.

13. The thirteenth part is a list of the author's awards and honors.

14. The fourteenth part is a list of the author's publications.

15. The fifteenth part is a list of the author's other activities.

16. The sixteenth part is a list of the author's other interests.

17. The seventeenth part is a list of the author's other achievements.

18. The eighteenth part is a list of the author's other contributions.

19. The nineteenth part is a list of the author's other works.

20. The twentieth part is a list of the author's other activities.

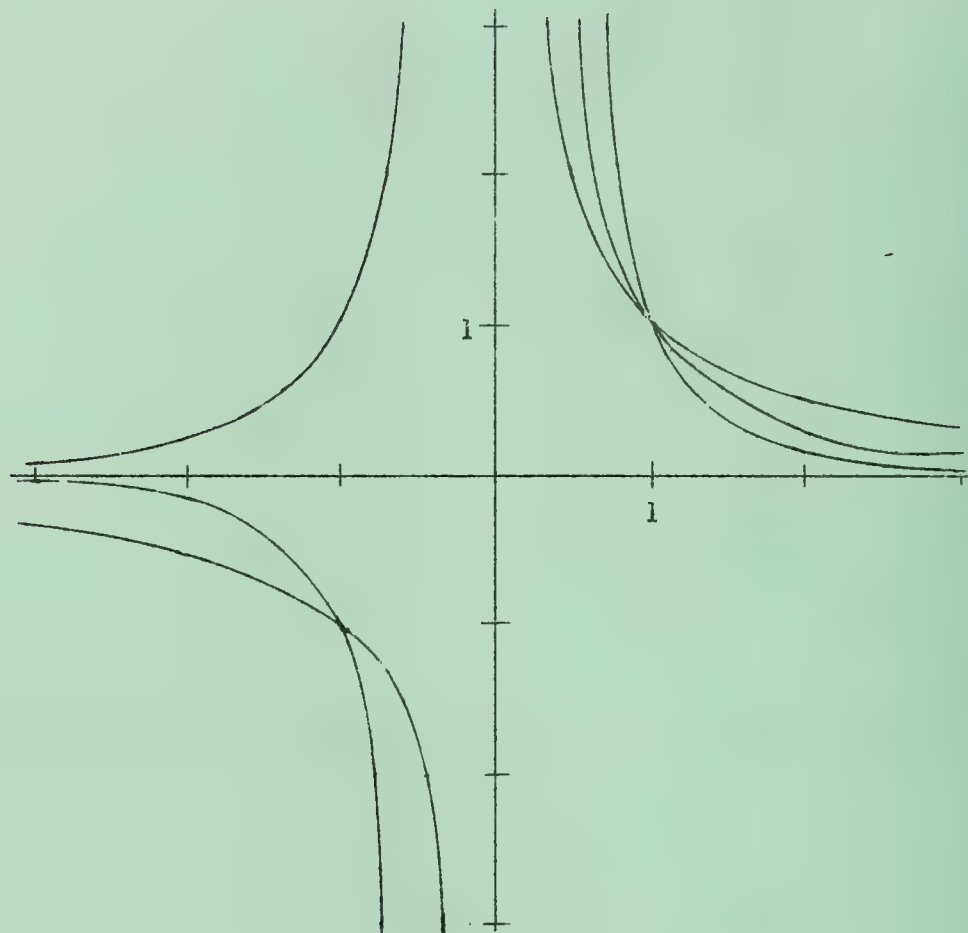
2. [The proof is similar to that for Exercise 1, but it uses Exercise 6 of Part G on page 8-103 and the theorem ' $\forall_{x \neq 0} 1/x \neq 0$ '. The latter follows [by contraposition] from Theorem 54 and the basic principle ' $1 \neq 0$ '.]
3. [As pointed out on page 8-116, the proof of this theorem is not difficult once one has (*) on page 8-116. In fact, the theorem of this exercise then follows from that of Exercise 1 exactly as that of Exercise 7 of Part G on page 8-103 follows from that of Exercise 5. For this proof, see COMMENTARY for Part G. Students who prove the theorem of the present exercise will probably follow the solution of Exercise 7 down to the sentence beginning 'Since $c^j \cdot c = c^{j+1}$,' and then interpolate a paraphrase of lines 9 through 17 on page 8-116.]

*

line 4 on page 8-116: For such a solution of Exercise 3, see the answer for Exercise 7 of Part G on page 8-103.

Proof of (**): Since $1 \in I$ it follows from Theorem 110c that $j - 1 \in I$. Hence, by ϵ^* , for $a \neq 0$, $a^{j-1+1} = a^{j-1}a$. Since $j - 1 + 1 = j$, it follows, for $a \neq 0$, by Theorem 49, that $a^{j-1} = a^j/a$. Consequently, $\forall_{x \neq 0} \forall_k x^{k-1} = x^k/x$.

4.



*

Answers for Part C.

1. By Theorem 86a, $k \geq 0$ or $k < 0$. Suppose that $k \geq 0$. Then for $a > 0$, by Exercise 5 of Part G on page 8-103, $a^k > 0$. Suppose that $k < 0$. Since, for $a > 0$, $a \neq 0$, it follows by definition that, for $a > 0$, $a^k = 1/a^{-k}$. Since $-k \in I^+$, it follows, again from Exercise 5, that $a^{-k} > 0$. Hence, by Theorem 99b, $1/a^{-k} > 0$. So, $a^k > 0$. Since, in both cases, for $a > 0$, $a^k > 0$, it follows that $\forall_{x>0} \forall_k x^k > 0$.

1870-1871

1871-1872

1872-1873

1873-1874

1874-1875

1875-1876

1876-1877

1877-1878

1878-1879

1879-1880

1880-1881

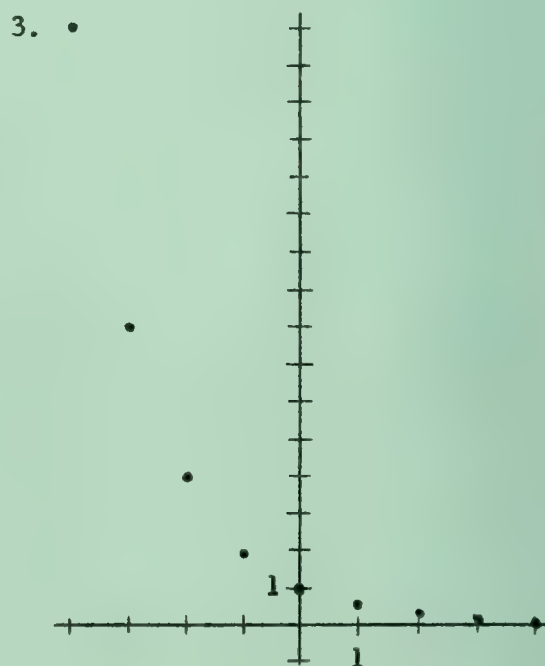
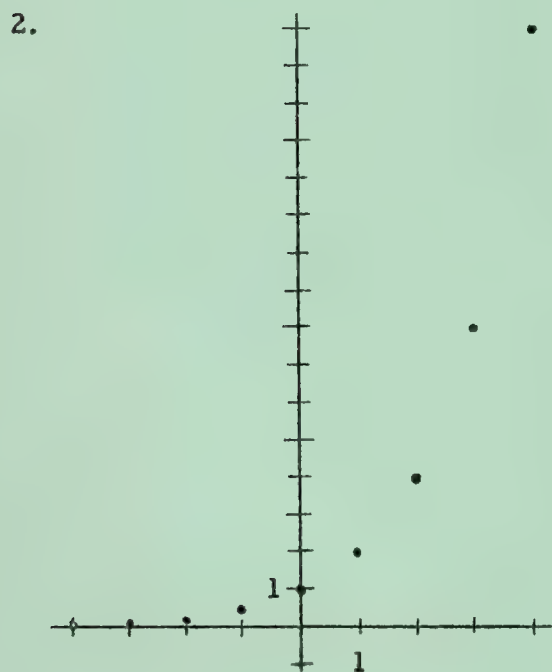
1881-1882

Answers for Part A.

1. $\frac{1}{9^4}$
2. $\frac{1}{5^{22}}$
3. $\frac{1}{7^5}$
4. 1
5. 2
6. $\frac{1}{3^2}$
7. $\frac{1}{6^6}$
8. $\frac{1}{9^{11}}$
9. 8
10. $\frac{1}{x}, [x \neq 0]$
11. $x, [x \neq 0]$
12. $\frac{x^2}{y}, [y \neq 0]$

*

Answers for Part B.



1940-1941

1940-1941

1940-1941

1940-1941

1940-1941

1940-1941

1940-1941

Part (iii):

$$(14) \quad \forall_{k \geq 0} a^j a^k = a^{j+k} \quad [(4), (13), \text{Th. 114}]$$

$$(15) \quad \forall_x \forall_{j \geq 0} \forall_{k \geq 0} x^j x^k = x^{j+k} \quad [(1) - (14)]$$

The reference ' (P_3) ' in the comment for step (13) refers to the fact that the restriction that $j + i$ be nonnegative is, by (P_3) , satisfied if $j \geq 0$ and $i \geq 0$.

Steps (6) and (10) are instances of (*) on page 8-116. Strictly, (*) should be included as a step in the proof. But, as we have omitted recursive definitions from proofs [giving only the appropriate instances] we here omit (*). Also, an instance of Theorem 110b is needed, strictly, to justify inferring (10) from (*).

Similar remarks apply to steps (13) and (17) on page 8-118.

lines 7 and 8 on page 8-117:

This remark can be expanded on by outlining part of a tree-diagram for part (ii) of the proof of Theorem 155. [In the following, the numerals refer to steps in the proof given on pages 8-117 and 8-118.]

$$\begin{array}{ccc}
 \begin{array}{c} * \\ (5) \\ \hline | \\ | \\ | \\ (12) \end{array} & & \begin{array}{c} * \\ (5) \\ \hline | \\ | \\ | \\ (19) \end{array} \\
 \hline
 (12) \text{ and } (19) \\
 \hline
 (5) \Rightarrow [(12) \text{ and } (19)] \\
 \hline
 (20) \quad *
 \end{array}$$

*

Remarks on the proof of Theorem 155:

The restrictions '[a ≠ 0]' are mostly necessitated by the fact that exponentiation for arbitrary integral exponents is defined only for nonzero bases. A difference theorem:

$$\forall_x \forall_{j \geq 0} \forall_{k \geq 0} x^j x^k = x^{j+k}$$

[See Exercise 2 of Part B on page 8-101] could be proved by replacing the five restrictions on page 8-117 by:

$$[j \geq 0], [j \geq 0, i \geq 0], [i \geq 0], [j \geq 0, i \geq 0], [j + i \geq 0]$$

respectively, changing the marginal comments for steps (6) and (10) from '[(*)]' to '[recursive definition]', and continuing from step (12) as follows:

$$\begin{array}{ll}
 (13) \quad \forall_{k \geq 0} [a^j a^k = a^{j+k} & \\
 \Rightarrow a^j a^{k+1} = a^{j+(k+1)}] & [j \geq 0] \quad [(5)-(12); (P_3)]
 \end{array}$$



lines 3 - 1 from bottom of page 8-119:

As with most cases of simplifying expressions, it is difficult to lay down a set of rules whereby one can tell when an expression has been "simplified". Moreover [since 'simpler' is always an elision for 'simpler for the present purpose'], it is quite unnecessary to do so. Any such rules would apply only to exercises in a given textbook. The important skill is to be able to manipulate expressions to obtain certain forms which are useful because they serve a definite purpose. [Examples: factoring to solve quadratic equations; manipulating exponential expressions to solve the equations at the bottom of page 8-123.] Our test [Unit 2] for simplicity which involves counting the number of indicated operations does not work too well here. For example, for both of the expressions:

$$x^{-2} \quad \text{and:} \quad \frac{1}{x^2}$$

there are two indicated operations. So, the expressions should be equally simple. Perhaps convention is such that it will not tolerate minus signs unless it has to! The answers given below represent just one person's view of what simplification of exponential expressions leads to. Be prepared to be tolerant of your students' interpretations. If a student expresses doubts about what "the final answer should be", tell him to put down several versions.

*

Answers for Part B.

- | | | | |
|---------------------------|--------------------------------|------------------------------------|---------------------|
| 1. 2^2 | 2. 1 | 3. $\frac{1+2^6}{2^3}$ [or: 8.125] | 4. $-\frac{1}{2^3}$ |
| 5. $\frac{1}{2^6}$ | 6. $\frac{1}{10^6}$ | 7. $\frac{1}{10^3}$ | 8. 1 |
| 9. $\frac{1}{5^8}$ | | | |
| 10. 2^2 | 11. $-\frac{1}{3^3}$ | 12. $\frac{100}{3}$ | 13. 1 |
| 14. $\frac{1}{2^4 3^2 5}$ | | | |
| 15. 2^3 | 16. $\frac{1}{2^{16} 3^3 5^2}$ | 17. $\frac{1}{5^2}$ | 18. 17 |
| 19. $\frac{1}{x^5}$ | | | |
| 20. $\frac{1}{y^6}$ | 21. x | 22. r | 23. a^{3-m} |
| 24. a^{2+k} | | | |
| 25. y^m | 26. $\frac{1}{k^2}$ | | |

[Answers for the rest of Part B are given in the COMMENTARY for page 8-121.]

Consequently, for $a \neq 0 \neq b$,

$$\forall_{k < 0} (ab)^k = a^k b^k.$$

4. For $b = 0$, $(a \div b)b = a$. For $a \neq 0 \neq b$, $a \div b \neq 0$. Hence, by Theorem 158, for $a \neq 0 \neq b$,

$$a^i = \left(\frac{a}{b} \cdot b\right)^i = \left(\frac{a}{b}\right)^i b^i.$$

Since, by Theorem 152b, for $b \neq 0$, $b^i \neq 0$, it follows from Theorem 49 that

$$\left(\frac{a}{b}\right)^i = \frac{a^i}{b^i}.$$

Consequently,

$$\forall_{x \neq 0} \forall_{y \neq 0} \forall_k \left(\frac{x}{y}\right)^k = \frac{x^k}{y^k}$$

[A slightly different theorem:

$$\forall_x \forall_{y \neq 0} \forall_{k \geq 0} \left(\frac{x}{y}\right)^k = \frac{x^k}{y^k}$$

can be proved in the same way, using the corresponding form of Theorem 158.] [An appropriate name for Theorem 159 is 'the distributive law for exponentiation over division'.]

5. [Two proofs for Theorem 160 are suggested by:

$$\begin{array}{ccccccc} \left(\frac{a}{b}\right)^{-i} & = & \frac{a^{-i}}{b^{-i}} & = & \frac{1/a^i}{1/b^i} & = & \frac{b^i}{a^i} = \left(\frac{b}{a}\right)^i \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ & \text{Th. 159} & \text{Th. 154} & \text{Th. 159} & & & \end{array}$$

and:

$$\begin{array}{ccccccc} \left(\frac{a}{b}\right)^{-i} & = & \frac{1}{(a/b)^i} & = & \frac{1}{a^i/b^i} & = & \frac{b^i}{a^i} = \left(\frac{b}{a}\right)^i \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ & \text{Th. 154} & \text{Th. 159} & \text{Th. 159} & & & \end{array}$$

]

*

$$(14) \quad = a^{j(i-1)} \quad [(13), \text{algebra}]$$

$$(15) \quad \forall_k [(a^j)^k = a^{jk} \Rightarrow ((a^j)^{k+1} = a^{j(k+1)} \\ \text{and } (a^j)^{k-1} = a^{j(k-1)})] \quad [(9), (14); *(4)]$$

Part (iii):

$$(16) \quad \forall_k (a^j)^k = a^{jk} \quad [(3), (15), \text{Th. 117}]$$

$$(17) \quad \forall_x \neq 0 \quad \forall_j \forall_k (x^j)^k = x^{jk} \quad [(1) - (16); \text{Th. 152b}]$$

[As in the case of the proof of Theorem 155 (see COMMENTARY for page 8-117) the preceding proof can be transformed into a proof of another theorem--in this case:

$$\forall_x \forall_j \geq 0 \quad \forall_k \geq 0 \quad (x^j)^k = x^{jk} \quad]$$

☆3. [We give the second part of the alternative proof suggested in the text. From the first part of the proof it follows that, for $a \neq 0 \neq b$,

$$\forall_k \geq 0 \quad (ab)^k = a^k b^k. \quad]$$

For $a \neq 0$ and $b \neq 0$, $ab \neq 0$. So, by definition, for $i < 0$ and $a \neq 0 \neq b$,

$$(ab)^i = \frac{1}{(ab)^{-i}}.$$

Since, for $i < 0$, $-i \geq 0$ and $-i \in I$, it follows from the theorem proved in the first part of the proof that, for $a \neq 0 \neq b$ and $i < 0$,

$$(ab)^{-i} = a^{-i} b^{-i};$$

whence

$$(ab)^i = \frac{1}{a^{-i} b^{-i}} = \frac{1}{a^{-i}} \cdot \frac{1}{b^{-i}}.$$

But, by definition, for $i < 0$, $a \neq 0$, and $b \neq 0$,

$$\frac{1}{a^{-i}} = a^i \text{ and } \frac{1}{b^{-i}} = b^i;$$

whence

$$(ab)^i = a^i b^i.$$

Answers for Part A.

1. [Compare with proof of (**) in the COMMENTARY for page 8-116.]

It follows from Theorem 110c that $j - k \in I$. Hence, by Theorem 155, for $a \neq 0$, $a^{j-k} a^k = a^{j-k+k}$. Since $j - k + k = j$ and since, by Theorem 152b, for $a \neq 0$, $a^k \neq 0$, it follows, by Theorem 49, that $a^{j-k} = a^j / a^k$. Consequently, $\forall x \neq 0 \forall_j \forall_k x^j / x^k = x^{j-k}$.

☆2. [Compare with proof of Theorem 155 on pages 8-117 and 8-118.]

Part (i):

- | | | | |
|-----|---------------|--------------|-----------------------------|
| (1) | $(a^j)^0 = 1$ | $[a \neq 0]$ | [recursive definition] |
| (2) | $= a^0$ | | [(1), recursive definition] |
| (3) | $= a^{j^0}$ | | [(2), algebra] |

Part (ii):

- | | | | |
|------|-------------------------------|--------------------------|-------------------------|
| (4) | $(a^j)^i = a^{ji}$ | $[a \neq 0]$ | [inductive hypothesis]* |
| (5) | $(a^j)^{i+1} = (a^j)^i a^j$ | $[a \neq 0, a^j \neq 0]$ | [(*)] |
| (6) | $= a^{ji} a^j$ | | [(4), (5)] |
| (7) | $a^{ji} a^j = a^{ji+j}$ | $[a \neq 0]$ | [Theorem 155] |
| (8) | $(a^j)^{i+1} = a^{ji+j}$ | | [(6), (7)] |
| (9) | $= a^{j(i+1)}$ | | [(8), algebra] |
| (10) | $(a^j)^{i-1} = (a^j)^i / a^j$ | $[a \neq 0, a^j \neq 0]$ | [(**)] |
| (11) | $= a^{ji} a^j$ | | [(4), (10)] |
| (12) | $a^{ji} / a^j = a^{ji-j}$ | $[a \neq 0]$ | [Theorem 156] |
| (13) | $(a^j)^{i-1} = a^{ji-j}$ | | [(10), (12)] |

There is a corresponding, more complicated, result for negative numbers:

$$\forall_{x < 0} \forall_j \forall_k [x^j = x^k \iff ([x = -1 \text{ and } k - j \text{ is even}] \text{ or } j = k)]$$

It is not particularly interesting because, in cases when one might use it, it is easier to use the technique used to prove it. However, here is a proof:

For $a < 0$, if $a^j = a^k$ then j and k are either both even or both odd and, in either case, $k - j$ is even. Also, for $a < 0$, $-a > 0$, $a^j = (-1)^j(-a)^j$ and $a^k = (-1)^k(-a)^k$. So, if $a^j = a^k$, $(-a)^j = (-1)^{k-j}(-a)^k$ and, since $k - j$ is even, $(-a)^j = (-a)^k$. Hence, by Theorem 161, $-a = 1$ or $j = k$. [Since if $j = k$ then $k - j$ is even, this completes the proof of the only if-part of the theorem.] On the other hand, if $a = -1$ and $k - j$ is even, or if $j = k$, $a^j = a^k$. Consequently, the theorem.

*

Complete the solution to the Sample of Part D by actually applying Theorem 161. Since $2 > 0$ and $3k + 5$ and -1 are integers,

$$2^{3k+5} = 2^{-1} \iff (2 = 1 \text{ or } 3k + 5 = -1).$$

But, $2 = 1$. So, $2^{3k+5} = 2^{-1} \iff 3k + 5 = -1$, and the latter is the case if and only if $k = -2$.

*

Answers for the exercises of Part D are given in the COMMENTARY for pages 8-123 and 8-124.

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Answers for Part C.

- | | | | |
|----------------------|--------------------|---------------------|-----------------------------|
| 1. x^{-2} | 2. x^{-7} | 3. $a^{-1}b^2$ | 4. $5a^{-1}y^3$ |
| 5. mnp^{-1} | 6. $xy^{-1}z^{-1}$ | 7. 2^{-n} | 8. 4^{-m} |
| 9. $x^{-2}y^2z^{-2}$ | 10. $5s^{-3}t^3$ | 11. $3x^3y^4$ | 12. $-9p^4q^6$ |
| 13. $48x^3y^3$ | 14. $2^4a^6b^2$ | 15. $2ab(a+b)^{-1}$ | 16. $(2b)^{-1} + (2a)^{-1}$ |

*

Here is a teaching approach to help students discover Theorem 161, especially the ' $x = 1$ ' part.

Teacher: I am thinking of a positive number. Let's suppose that John gives me an exponent [John whispers one to the teacher] and I raise my positive number to that power. Next, suppose that Mary gives me an exponent [she whispers] and again I raise my original number to Mary's power. It turns out that the results are equal. What can you tell me about the exponents Mary and John picked?

Student: They are equal.

Teacher: Well, they might have been equal. But, actually they weren't. Yet, I did get the same result. How could that be?

Student: Aha, you were thinking of 0!

Teacher: Pretty clever, but remember I said that I was thinking of a positive number. But, you're on the way, now.

Student: You were thinking of 1.

Teacher: Right.

*

27. $\frac{1}{y^4}$

28. $\frac{1}{z^5}$

29. r^2

30. $\frac{1}{r^2}$

31. a^7

32. n^8

33. $\frac{1}{m^8}$

34. $z^3 + m$

35. t^{2m}

36. x^{5m}

37. $\frac{1}{5}$

38. m^3

39. $\frac{a^5}{b^2}$

40. t

41. $\frac{2}{x^6 y^7}$

42. $c^2 k^3$

43. $\frac{s^2}{3}$

44. $\frac{n^3}{5}$

45. $\frac{3y^2}{4x^3}$

46. $\frac{2v^5}{3u^2}$

47. $\frac{b^5}{a^5}$

48. $\frac{x^6}{y^7}$

49. $\frac{r^5 t^9}{s^7}$

50. $\frac{b^5}{a^4}$

51. $\frac{2b^2}{a^4}$

52. $\frac{5x^4 - 3}{x^2}$

53. $5yz$

54. $s^3 + t^3$

55. $\frac{5^5 y^4}{x^2 z^7}$

56. $\frac{2^3 \cdot 3b}{a^8 c^3}$

57. $\frac{4xz^5}{3^2 y^5}$

58. $\frac{2^3 x}{3^2 y^4}$

59. $\frac{x^{18} y^{11}}{3^{11}}$

60. $\frac{4}{xy}$

61. $\frac{1 + 3(a + b)}{(a + b)^2}$

62. $\frac{x^6 z^4}{y^4}$

63. $\frac{x^6 + 1}{x^5 + x}$

64. 1

65. $xz + yz + 1$

66. $\frac{x}{z + y}$

67. $\frac{x^2 y z^4 + x^2 z^3}{y^4}$

68. $-x^2 y^3$

69. $\frac{xa}{a - x}$

70. $u^2 v^2$

71. $\frac{1 - 3^6}{3^7}$

72. $\frac{1}{x^2 y^2}$

73. $y^k + x^k$

74. $-xy(x + y)$

75. $\frac{x^k y^k + 1}{y^k}$

76. $\frac{x^k}{1 - x^k}$

77. $\frac{2^m - 2^n \cdot 3^m x^{2m - 5n} y^{4m + n}}{5^n}$

78. $\left(\frac{5 \cdot 7^3 s^{23}}{3t^9}\right)^j$

79. $x^p - 2q$

80. $\frac{1}{a^4 j + 1}$

81. 1

82. 1

83. $b^{n+p} c^{p+m} a^{m+n}$

The value of the foregoing discussion is that it motivates Bernoulli's Inequality which is given in Exercise 2(f) on page 8-125. Students should be led at this time to formulate their discovery by:

$$(*) \quad \forall_x \forall_{k \geq 0} (1+x)^k \geq 1+kx$$

This generalization is false, as one can easily tell by choosing a value for 'x' such that the corresponding value of '1+x' is negative and choosing an odd value for 'k'. For example, the instance:

$$(1 + -5)^3 \geq 1 + 3 \cdot -5$$

is false. This suggests a restriction on the domain of 'x'. Students might restrict more than necessary and suggest the following as a modification of (*):

$$\forall_{x > 0} \forall_{k \geq 0} (1+x)^k \geq 1+kx$$

You can then counter this proposal with:

$$\forall_{x \geq 0} \forall_{k \geq 0} (1+x)^k \geq 1+kx,$$

and, after allowing a moment for checking, propose the following:

$$\forall_{x \geq -1} \forall_{k \geq 0} (1+x)^k \geq 1+kx$$

Since this is Bernoulli's Inequality, students will not be able to find a counter-example. You can let the matter drop at this time and move on to Exercise 2, or you can have the students prove Bernoulli's Inequality by induction [that is, solve Exercise 2(f)] and then move immediately to Exercise 2☆(g) or to Exercise 3. [The earlier parts of Exercise 2 do have value in that they stimulate geometric intuition, and will help students in doing Exercise 4.]

*

2. [The function in question is called a two-way sequence because its domain, I, "stretches both ways".]

Answers for parts (a) - (g) of Exercise 2 are given in the COMMENTARY for page 8-125.

2. The required digit is '9'. [$2^5 n^2 = 2500 + 10n + 2$]

*

Answers for Part F.

1. This problem is just a kickoff for Exercise 3. It is designed to make students aware of the fact that the terms of an exponential sequence with base greater than 1 [even just a tiny bit greater] get larger and larger [even though very slowly].

| | |
|---------------------------------|-------------------------|
| (a) $1.0001^0 = 1$ | [first term, $k = 0$] |
| $1.0001^1 = 1.0001$ | [second term, $k = 1$] |
| $1.0001^2 = 1.00020001$ | [third term, $k = 2$] |
| $1.0001^3 = 1.000300030001$ | [fourth term, $k = 3$] |
| $1.0001^4 = 1.0004000600040001$ | [fifth term, $k = 4$] |

(b) 'Yes', to all three questions. Exercise 3 (a) takes care of this problem, but intuition alone can be quite helpful here. By observing how the multiplication algorithm worked in getting the terms in part (a) or by just studying the pattern of the terms, one guesses that

| | | |
|----------------|---|---------|
| 1.0001^9 | > | 1.0009, |
| 1.0001^{10} | > | 1.001 , |
| 1.0001^{99} | > | 1.0099, |
| 1.0001^{100} | > | 1.01 . |

So, it seems likely that

| | | |
|--------------------|---|-------|
| 1.0001^{1000} | > | 1.1 , |
| 1.0001^{10000} | > | 2 , |
| 1.0001^{100000} | > | 11 , |
| $1.0001^{1000000}$ | > | 101 . |

It looks as if the 1000001st term is greater than 101. Of course, this does not mean that this is the first one of the terms which is greater than 101.]

*

1. 1/2

And the first part of the book is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \int_0^x f(t) dt + \int_0^x f(t) dt + \int_0^x f(t) dt + \dots$$

It is easy to see that the function $f(x)$ is continuous and that it satisfies the equation

$$f(x) = \int_0^x f(t) dt + \int_0^x f(t) dt + \int_0^x f(t) dt + \dots$$

It is also easy to see that the function $f(x)$ is differentiable and that its derivative is given by the equation

$$f'(x) = f(x) + f(x) + f(x) + \dots$$

It is easy to see that the function $f(x)$ is analytic and that its power series expansion is given by the equation

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Answers for Part D [which begins on page 8-122].

1. 1 $[16 = 2^4]$
2. 9 $\left[\frac{1}{3} = 3^{-1}\right]$
3. Each integer is a root. [Theorem 150a]
4. no roots [Theorem 150a and ' $1 \neq 2$ ']
5. -5 $[4^{k-1} = 2^{2(k-1)} \text{ and } 8^{1+k} = 2^{3(1+k)}]$
6. -5 [Compare with Exercise 5.]
7. -1 $[3^{k+1} 10^{k+1} = 9^{k+1} 10^{k+1} \iff 3^{k+1} = 3^{2(k+1)}$
(Theorem 152b)]
8. $(-3, 2)$ $[2k = 1 - j, 3k = 9 + j]$
9. $(2, 1)$ $[i - 2 - (j - 1) = 0, i - 2 = -(j - 1)]$
10. -2, 2 $[(2^k)^2 + 1 = \frac{17}{4} \cdot 2^k \text{--a quadratic equation in '2^k'}]$
- ★11. no roots $[\forall_k (2^k + 2^{-k})^2 = 4^k + 2 + 4^{-k}]$
- ★12. -2, -1, 1, 2 $[2^k + 2^{-k} = x, 8(x^2 - 2) - 54x + 101 = 0]$

*

Answers for Part ★E.

1. For each x ,

| | | |
|-------|-------|-------|
| x^2 | x^7 | x^6 |
| x^9 | x^5 | x^1 |
| x^4 | x^3 | x^8 |

is a magic multiplication-square.

The product of the numbers listed in each row, in each column, and in each diagonal is x^{15} . [If you wish to restrict magic multiplication-squares to arrays of different numbers then restrict x such that $0 \neq x^2 \neq 1$.]

$$\begin{aligned}
f_{j+2} f_j - f_{j+1}^2 &= (f_j + f_{j+1}) f_j - (f_{j-1} + f_j)^2 \\
&= f_j^2 + f_{j+1} f_j - (f_{j-1}^2 + 2f_{j-1} f_j + f_j^2) \\
&= f_{j+1} f_j - f_{j-1}^2 - 2f_{j-1} f_j \\
&= (f_{j+1} f_j - f_{j-1} f_j) - (f_{j-1}^2 + f_{j-1} f_j) \\
&= (f_{j+1} - f_{j-1}) f_j - f_{j-1} (f_{j-1} + f_j) \\
&= f_j f_j - f_{j-1} f_{j+1} \\
&= -(f_{j-1} f_{j+1} - f_j^2) \\
&= -(-1)^j = (-1)^{j+1}.
\end{aligned}$$

Hence, $\forall_{k \geq 0} [f_{k+1} f_{k-1} - f_k^2 = (-1)^k \Rightarrow f_{k+2} f_k - f_{k+1}^2 = (-1)^{k+1}.$

(iii) From (i) and (ii) it follows, by Theorem 114, that

$$\forall_{k \geq 0} f_{k+1} f_{k-1} - f_k^2 = (-1)^k.$$

2. The extra square unit of area in the second figure is accounted for by a parallelogram-shaped hole in the second figure. [One diagonal of the parallelogram is a diagonal of the rectangle. The other joins the vertices of the obtuse angles of the two trapezoids.] Since the tangent ratio of the acute angle of either trapezoid is $5/2$, while that of the larger acute angle of either triangle is $8/3$, and since $5/2 < 8/3$, it is clear that, for example, in the second figure, the slanting leg of the upper trapezoid slants up more steeply than does the hypotenuse of the upper right triangle.

The answer for Exercise 3 of Part C is given in the COMMENTARY for page 8-127.

$$\left(\frac{1}{b}\right)^P > \frac{1}{c} \iff \left(\frac{1}{b}\right)^P (b^P c) > \frac{1}{c} (b^P c).$$

Hence, using Theorem 158 [as well as the apm, cpm, and the pq], it follows that

$$\left(\frac{1}{b}\right)^P > \frac{1}{c} \iff b^P < c.$$

So, for $b \neq 1$ and $c > 0$,

$$\text{if } \left(0 < b < 1 \text{ and } p \geq \frac{1}{c(1-b)}\right) \text{ then } b^P < c.$$

Consequently, Theorem 165.

- (d) The $(10^{10} + 1)$ th term is less than 10^{-6} . [This term is $0.9999^{10^{10}}$.]
 [The p th term is 0.9999^{P-1} and, by Theorem 165, this is less than 10^{-6} if $p - 1 \geq \frac{1}{10^{-6} 10^{-4}}$.];

The $(4375961 \cdot 10^4 + 1)$ th term is less than $1/4375961$.

(e) yes; yes

- (f) [This is impossible. By Theorem 152a, for $b > 0$, b^P is positive for each p . So, no term is less than (or even equal to) a non-positive number.]

*

Answers for Part ☆G.

1. (i) Since $f_1 = f_{-1} + f_0 = 1$, it follows that $f_{0+1} f_{0-1} - f_0^2 = 1 \cdot 1 - 0^2 = 1 = (-1)^0$.

- (ii) Suppose [for a $j \geq 0$] that $f_{j+1} f_{j-1} - f_j^2 = (-1)^j$. It follows from the recursive definition that [since $j+1 \in I^+$ and $j+2 \in I^+$]

$$\begin{aligned}
\frac{1}{c} > 1 &\iff \frac{1}{c} \cdot c^2 > 1 \cdot c^2 \\
&\iff c^2 - c < 0 \\
&\iff c(c - 1) < 0 \\
&\iff [(c > 0 \text{ and } c < 1) \text{ or } (c < 0 \text{ and } c > 1)].
\end{aligned}$$

Since $1 \neq 0$, it is not the case that $c < 0$ and $c > 1$. Hence, for $c \neq 0$,

$$\frac{1}{c} > 1 \iff (c > 0 \text{ and } c < 1).$$

Consequently,

$$\forall_{x \neq 0} \left[\frac{1}{x} > 1 \iff 0 < x < 1 \right].$$

- (c) Suppose, for $b \neq 1$ and $c > 0$, that $p \geq \frac{1}{c(1-b)}$ and that $0 < b < 1$. Since $b > 0$ and $0 \neq 0$, it follows that $b \neq 0$ and, so, by Theorem 164, that, since $0 < b < 1$, $1/b > 1$. Consequently, by Theorem 163, it follows [for $c \neq 0$ and $0 \neq b \neq 1$] that

$$\text{if } p \geq \frac{1/c}{1/b - 1} \text{ then } \left(\frac{1}{b}\right)^p > \frac{1}{c}.$$

Now [for $c \neq 0 \neq b \neq 1$], $\frac{1/c}{1/b - 1} = \frac{b}{c(1-b)}$ and, since $b < 1$, it follows [for $c > 0$] that $c(1-b) > 0$. Hence, since $b < 1$, $\frac{b}{c(1-b)} < \frac{1}{c(1-b)}$. So,

$$\frac{1}{c(1-b)} > \frac{1/c}{1/b - 1},$$

and, since $p \geq \frac{1}{c(1-b)}$, it follows that $p \geq \frac{1/c}{1/b - 1}$. Hence, [for $b \neq 1$ and $c > 0$, and assuming that $0 < b < 1$ and $p \geq \frac{1}{c(1-b)}$], $\left(\frac{1}{b}\right)^p > \frac{1}{c}$.

Since, by hypothesis, $b > 0$, it follows from Theorem 152a that $b^p > 0$. So [for $c > 0$], $b^p c > 0$ and, by the mtpi [for $b \neq 0 \neq c$],

sufficiently large, $1.0001^p > 1/c$, it should be possible to prove that, for p this large, $0.9999^p < c$. [This insight can lead to a proof of Theorem 165 different from that suggested in the Hint for part (c).] Since the only relevant property of the base 0.9999 is that it is less than 1 and positive it should be possible to prove a similar result for any exponential sequence whose base is between 0 and 1. [Actually, as some students may see, it is sufficient that the base be between -1 and 1--but, for our purpose, it is sufficient to restrict our considerations to exponential sequences with positive bases. So, we wish, for $0 < b < 1$, to derive a conclusion of the form:

$$\dots \Rightarrow b^p < c$$

whose antecedent places some condition of largeness on p . This suggests Theorem 163, but has the inequality sign reversed. But, this is to be expected, since, if $0 < b < 1$, $1/b > 1$ [Theorem 164, to be proved in part (c)], and, by Theorem 163, it follows that, for p sufficiently large, $(1/b)^p > 1/c$. It is not difficult to see from this that, for p sufficiently large]and $c > 0$], $b^p < c$. [From Theorem 163, we see that p is sufficiently large if

$$p \geq \frac{\frac{1}{c}}{\frac{1}{b} - 1}$$

--that is [for $c \neq 0 \neq b \neq 1$], if

$$p \geq \frac{b}{c(1 - b)}.$$

And, since $b < 1$ and $c(1 - b) > 0$, the simpler requirement that

$$p \geq \frac{1}{c(1 - b)}$$

is sufficient.] The preceding argument, leading to Theorem 165, is presented in somewhat more detail in the answers to parts (b) and (c).

(b) For $c \neq 0$, $c^2 > 0$ and

The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function, and its value is determined by the initial condition $f(0) = 1$.

In the second part, we consider the problem of finding the maximum value of the function $f(x)$ on the interval $[0, 1]$. It is shown that the maximum value is attained at $x = 0$ and is equal to 1.

The third part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function, and its value is determined by the initial condition $f(0) = 1$.

In the fourth part, we consider the problem of finding the maximum value of the function $f(x)$ on the interval $[0, 1]$. It is shown that the maximum value is attained at $x = 0$ and is equal to 1.

The fifth part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function, and its value is determined by the initial condition $f(0) = 1$.

In the sixth part, we consider the problem of finding the maximum value of the function $f(x)$ on the interval $[0, 1]$. It is shown that the maximum value is attained at $x = 0$ and is equal to 1.

The seventh part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function, and its value is determined by the initial condition $f(0) = 1$.

In the eighth part, we consider the problem of finding the maximum value of the function $f(x)$ on the interval $[0, 1]$. It is shown that the maximum value is attained at $x = 0$ and is equal to 1.

The ninth part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function, and its value is determined by the initial condition $f(0) = 1$.

In the tenth part, we consider the problem of finding the maximum value of the function $f(x)$ on the interval $[0, 1]$. It is shown that the maximum value is attained at $x = 0$ and is equal to 1.

(c) The $\left(\left\lfloor \frac{c-1}{b-1} \right\rfloor + 2\right)$ th term is greater than c . [This term is

$b^{\left\lfloor \frac{c-1}{b-1} \right\rfloor + 1}$.] $[b^{k-1} = [1 + (b-1)]^{k-1} \geq 1 + (b-1)(k-1) > c$ if $k-1 > \frac{c-1}{b-1}$. By Theorem 118b, this is the case if and only if $k-1 > \left\lfloor \frac{c-1}{b-1} \right\rfloor$, and this is the case if and only if $k \geq \left\lfloor \frac{c-1}{b-1} \right\rfloor + 2$.]

(d) Suppose, for $b > 1$, that $p \geq \frac{c}{b-1}$. Since $c > c-1$ and $b-1 > 0$, it follows that $c/(b-1) > (c-1)/(b-1)$. So, $p > (c-1)/(b-1)$. Now, by Bernoulli's Inequality [as in part (c), above],

$$b^p > c \text{ if } p > \frac{c-1}{b-1}.$$

Since the latter is the case, $b^p > c$. Hence, for $b > 1$,

$$\forall_n [n \geq \frac{c}{b-1} \Rightarrow b^n > c].$$

Consequently,

$$\forall_{x>1} \forall_y \forall_n [n \geq \frac{y}{x-1} \Rightarrow x^n > y].$$

4. (a) This part of the exercise should provide motivation for the rest of it. [For your information, Theorem 165 is needed for the discussion, beginning on page 8-138, of infinite geometric progressions. Theorem 163 is needed in Unit 9.] The students' discovery in Exercise 2(d) should lead them to believe that the terms of the sequence in question decrease in such a way that, for any positive number c , however small, there is a p such that $0.9999^p < c$. In fact, $0.9999 < 1/1.0001$ [$\forall_{0 \neq x > -1} 1-x < 1/(1+x)$ --an easily proved, and useful theorem] and, although this has not been proved [but, see Exercise ☆23 on page 8-222], one would suspect that it follows that, for any p , $0.9999^p < (1/1.0001)^p$. And since we know from Exercise 3(d) that, for p

Combining this last result and the second alternative for part (ii) we see that, using Theorem 114, we may infer, for $0 \neq c \geq -1$:

$$\forall_{k \geq 2} (1 + c)^k > 1 + kc$$

In particular, we may infer this for $c > 0$. Hence,

$$\forall_{x > 0} \forall_{k \geq 2} (1 + x)^k > 1 + kx,$$

and this is, since $k \geq 2$ if and only if $k > 1$, equivalent to the theorem in question.

3. (a) The k th term of the exponential sequence with base 1.0001 is 1.0001^{k-1} . So, we want a number k such that

$$1.0001^{k-1} > 1000001.$$

By Bernoulli's inequality,

$$1.0001^{k-1} \geq 1 + 0.0001(k-1).$$

So, it is sufficient to find a number k such that

$$1 + 0.0001(k-1) > 1000001.$$

This last is the case if and only if $k-1 > 10^{10}$. The smallest k for which this is the case is $10^{10} + 2$. So, the $(10^{10} + 2)$ th term of the sequence is greater than 1000001. [This term is $(1.0001)^{10^{10}+1}$.]

[Bernoulli's Inequality gives only a very rough estimate. Actually, the first term greater than 1000001 is the 138250th. In fact, a short logarithmic calculation shows that $1.0001^{138249} > 1000010$ and another shows that $1.0001^{138248} < 1000000$.]

- (b) The $(3 \cdot 10^{10} - 1)$ th term is greater than 10^{10} . [This term is

$$\left(\frac{4}{3}\right)^{3 \cdot 10^{10} - 2} .] \left[\left(\frac{4}{3}\right)^{k-1} = \left(1 + \frac{1}{3}\right)^{k-1} \geq 1 + \frac{k-1}{3} > 10^{10} \text{ if}$$

$k-1 > 3 \cdot (10^{10} - 1)$. This is the case if $k-1 = 3 \cdot 10^{10} - 2$.]

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Inequality but having ' $>$ ' in place of ' \geq '. This is such a theorem. Its proof, by induction, throws a little stronger light on the role of the restrictions in Theorem 162.]

The proof is much like that of Theorem 162 and can best be appreciated by studying the proof of that theorem with an eye to improving on it. Looking at part (ii) in the answer for part (f), we see that, starting with the assumption:

$$(1 + c)^j > 1 + jc$$

we may, for $1 + c > 0$, infer:

$$(1 + c)^{j+1} > (1 + jc)(1 + c)$$

Then, as before, for $j \geq 0$, $jc^2 \geq 0$ and, so, we infer:

$$(1 + jc)(1 + c) \geq 1 + (j + 1)c$$

Hence, for $c > -1$,

$$\forall_{k \geq 0} [(1 + c)^k > 1 + kc \Rightarrow (1 + c)^{k+1} > 1 + (k + 1)c],$$

by Theorem 92.

Alternatively, for $1 + c \geq 0$, we may, from the inductive hypothesis, infer:

$$(1 + c)^{j+1} \geq (1 + jc)(1 + c)$$

Then, for $j > 0$ and $c \neq 0$, $jc^2 > 0$ and, so, we infer:

$$(1 + jc)(1 + c) > 1 + (j + 1)c$$

Hence, for $0 \neq c \geq -1$,

$$\forall_{k \geq 1} [(1 + c)^k > 1 + kc \Rightarrow (1 + c)^{k+1} > 1 + (k + 1)c].$$

Having found two ways of getting ' $>$ ' in place of ' \geq ' in part (ii) of the proof, we now look at part (i). Since, for any c , $(1 + c)^0 = 1 + 0c$ and $(1 + c)^1 = 1 + 1c$, we cannot start our induction before 2. But, $(1 + c)^2 = 1 + 2c + c^2$ and, so, by Theorem 97a, $(1 + c)^2 > 1 + 2c$ for $c \neq 0$.

1. The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \sum_{n=0}^{\infty} \frac{f_n(x)}{n!}$$

$$f_n(x) = \frac{1}{n!} \left(\frac{d}{dx} \right)^n f(x)$$

where $f(x)$ is a function of the class \mathcal{C}^∞ .

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For this, one uses Theorem 116 ["backward induction"]. Part (i) of the proof is as above. Part (ii) proceeds as follows:

Suppose that [for a $j \leq 0$ and a $c > -1$]

$$(1 + c)^j \geq 1 + jc.$$

Then, since $1 + c > 0$,

$$\frac{(1 + c)^j}{1 + c} \geq \frac{1 + jc}{1 + c}$$

and, so,

$$(1 + c)^{j-1} \geq \frac{1 + jc}{1 + c}$$

But, since $j \leq 0$ and $1 + c > 0$,

$$\frac{1 + jc}{1 + c} \geq 1 + (j - 1)c.$$

For, by the mtpi, this is the case if and only if

$$\frac{1 + jc}{1 + c} (1 + c) \geq [1 + (j - 1)c](1 + c)$$

--that is, if and only if

$$1 + jc \geq 1 + jc + (j - 1)c^2.$$

And this is the case since $(j - 1)c^2 \leq 0$ [$c^2 \geq 0$ and $j - 1 \leq 0$].

Hence, for $c > -1$,

$$\forall_{k \leq 0} [(1 + c)^k \geq 1 + kc \Rightarrow (1 + c)^{k-1} \geq 1 + (k - 1)c].$$

The remainder of the proof [part (iii) and a generalizing step] is trivial. [Notice that (***) cannot be proved by using Theorem 117, since we cannot justify the quantifier ' \forall_k ' which appears in the antecedent of the latter theorem. Of course, one could formulate a more complicated theorem than Theorem 117 which would do the job.]

☆(g) [Sometimes (but not in this text) one needs a theorem like Bernoulli's

TC[8-125, 126]d

and the other two are the same as in the previous case. The only difference is that the first two are now the same as the last two. This is because the first two are now the same as the last two. This is because the first two are now the same as the last two.

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given below.] For most purposes [see, for example, Theorems 163 and 165] one need consider only positive values of 'k'. For these, and even for $k \geq 0$, one can relax slightly the conditions ' $x > -1$ '. So, we arrive at Bernoulli's Inequality:

$$\forall_{x \geq -1} \forall_{k \geq 0} (1+x)^k \geq 1+kx$$

Proof of Bernoulli's Inequality:

(i) Since $(1+c)^0 = 1$ and $1+0c = 1$, $(1+c)^0 \geq 1+0c$.

(ii) Suppose that [for a $j \geq 0$ add a $c \geq -1$]

$$(1+c)^j \geq 1+jc.$$

Then, since $1+c \geq 0$,

$$(1+c)^{j+1} \geq (1+jc)(1+c)$$

and, so,

$$(1+c)^{j+1} \geq (1+jc)(1+c) = 1+(j+1)c+jc^2.$$

But, since $j \geq 0$ and $c^2 \geq 0$ [Theorem 97a and the pm0], $jc^2 \geq 0$ and

$$1+(j+1)c+jc^2 \geq 1+(j+1)c.$$

Hence, for $c \geq -1$,

$$\forall_{k \geq 0} [(1+c)^k \geq 1+kc \Rightarrow (1+c)^{k+1} \geq 1+(k+1)c].$$

(iii) From (i) and (ii) it follows, for $c \geq -1$, by Theorem 114, that

$$\forall_{k \geq 0} (1+c)^k \geq 1+kc.$$

Consequently,

$$\forall_{x \geq -1} \forall_{k \geq 0} (1+x)^k \geq 1+kx.$$

To prove (***) it is sufficient, now, to prove:

$$\forall_{x > -1} \forall_{k \leq 0} (1+x)^k \geq 1+kx$$

$$f(t) = \frac{1}{2} (1 + \cos \pi t) \quad (1)$$

where $f(t)$ is the function defined by (1) and π is the constant π .

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Now,

$$\{(k, y): y = (b - 1)k + 1\} \subseteq \{(x, y): y = (b - 1)x + 1\}.$$

So, our observation in part (e) can be expressed by:

$$\forall_{b > 1} \forall_k b^k \geq (b - 1)k + 1$$

or, as in the text:

$$(*) \quad \forall_{b > 1} \forall_k b^k \geq 1 + (b - 1)k$$

This theorem can be proved by induction. However, its only purpose is to suggest, by geometric considerations, Bernoulli's Inequality. [The latter theorem has already been suggested by the arithmetic considerations of Exercise 1.] In fact, it follows from (*) that, for $1 + c > 1$ [that is, for $c > 0$],

$$(1 + c)^j \geq 1 + [(1 + c) - 1]j = 1 + jc.$$

So, from (*),

$$(**) \quad \forall_{x > 0} \forall_k (1 + x)^k \geq 1 + kx.$$

Similarly, (*) is a consequence of (**).

If one repeats part (e) for the graph constructed for part (d)--instead of for the graph given on page 8-124 [again labelling by 'Q' the point whose x-coordinate is 1]--a repetition of the preceding discussion leads one to generalizations like (*) and (**), but with ' $\forall_{b > 1}$ ' and ' $\forall_{x > 0}$ ' replaced by ' $\forall_{0 < b < 1}$ ' and ' $\forall_{-1 < x < 0}$ '. Since

$$(1 + 0)^j = 1 \geq 1 = 1 + j0,$$

we can guess that

$$(***) \quad \forall_{x > -1} \forall_k (1 + x)^k \geq 1 + kx.$$

This is, in fact, a theorem and can be proved by using Theorem 116.

[The proof is an extension of the proof of Bernoulli's Inequality which is

$$\{(x, y) \mid y = (x - 1)^2\}$$

(i) From part (a) we know that $f(x) = (x - 1)^2$.

are of:

names a variable quantity and may be used as if it were a number.
 $y(x) = f(x) = (x - 1)^2$ where $f(x)$ is a function, x is a variable.
 (a) slope of $f(x) = \frac{f(x) - f(0)}{x - 0} = \frac{(x - 1)^2 - (-1)}{x} = \frac{x^2 - 2x + 1 + 1}{x} = \frac{x^2 - 2x + 2}{x}$

of the y-axis.

be obtained by reflecting the graph in page 8-125 in the y-axis.
 follow that $\frac{1}{x} = \frac{1}{x}$. Hence, the graph of the desired kind can
 [Theorem 107, 108]. Since, in the case of reflection, $f(x) = 1/x$
 case follows from the fact that, since $f(x) = 1/x$, $f(1/x) = x$.
 of the graph of $y = 1/x$. That this means y is the
 page 8-125, the graph of $y = \frac{1}{x}$ is the reflection of the graph
 (b) As suggested by the answers for $f(x) = x^2$ and $f(x) = 1/x$ on

Yes, we can. If $f(x) = x^2$, then $f(x) = x^2$ for $x > 0$.

[Theorem 107]. But $y(x) = x^2$ for $x > 0$.

(c) The same would be true for exponential and logarithmic functions.

Since $f(x) = x^2$, it follows that $f(x) = x^2$ for $x > 0$.

For $x < 0$, $f(x) = x^2$ for $x < 0$, then $f(x) = x^2$ for $x < 0$.

Yes, we can. If $f(x) = x^2$ for $x > 0$, then $f(x) = x^2$ for $x > 0$.

negative exponents are not defined.

we can get it in the second quadrant since $f(x) = x^2$ for $x < 0$.

For $x < 0$, $f(x) = x^2$ for $x < 0$, then $f(x) = x^2$ for $x < 0$.

(b) If the same function $f(x) = x^2$ would be $f(x) = x^2$ for $x < 0$.

2. (a) (i) $\{x \mid x > 0\}$

Correction: On page 8-125 the underlining of 'Bernoulli's Inequality' should be moved from line 12b to line 16b.

2. (a) $(0, 1) \{ \forall_b b^0 = 1 \}$

(b) If the base were 0, the coordinates of Q would be $(1, 0)$ since $0^1 = 0$. But $y(Q) \neq 0$. So, $b \neq 0$. [Also, if $b = 0$, there would not be points in the second quadrant since powers of 0 with negative exponents are not defined.]

Yes, we can tell [if we can trust the picture]. If b were negative, some of the points--Q for example--would have negative y-coordinates. But, $y(Q) \neq 0$. So, the base is nonnegative. Since $b \neq 0$, it follows that b is positive.

(c) If the base were 1, the exponential sequence would be a constant [Theorem 150a]. But $y(P) \neq y(Q)$. So, $b \neq 1$.

Yes, we can tell. Since $b = b^1 = y(Q) > y(P) = 1$, $b > 1$.

(d) As suggested by the answers for Exercises 2 and 3 of Part B on page 8-115, the graph of $y = \left(\frac{1}{b}\right)^k$ is the reflection, in the graph of the y-axis, of the graph of $y = b^k$. That this actually is the case follows from the fact that, since $b \neq 0$, for each k , $\left(\frac{1}{b}\right)^k = b^{-k}$ [Theorems 160 and 50]. Since, in the case in question, $b > 1$, it follows that $0 < \frac{1}{b} < 1$. Hence, a graph of the required kind can be obtained by reflecting the one given on page 8-124 in the graph of the y-axis.

(e) slope of $\overleftrightarrow{PQ} = \frac{y(Q) - y(P)}{x(Q) - x(P)} = \frac{b^1 - b^0}{1 - 0} = b - 1$; y-intercept of $\overleftrightarrow{PQ} = y(P) = b^0 = 1$ [' $x(Q)$ ' rather than ' $k(Q)$ ' for in such contexts, ' x ' names a variable quantity and may be read as 'the first coordinate of'.]

(f) From part (e) we know that \overleftrightarrow{PQ} is the graph of

$$\{(x, y): y = (b - 1)x + 1\}.$$

1. The first part of the paper discusses the importance of the study and the objectives of the research.

2. The second part of the paper describes the methodology used in the study and the data collection process.

3. The third part of the paper presents the results of the study and discusses the implications of the findings.

is a solution of the given recursion equation. To satisfy the initial conditions stated in the recursive definition, x and y must be such that $x - 2y = 0$ and $x + y = 1$. So, $x = \frac{2}{3}$ and $y = \frac{1}{3}$. Hence, the answer.]

4. $\forall k \geq -1 \ a_k = \frac{1}{3} [(2a_0 + a_{-1}) + (a_0 - a_{-1}) \frac{(-1)^k}{2^k}]$ [The work is the same as that for Exercise 3, except that x and y must be such that $x - 2y = a_{-1}$ and $x + y = a_0$. So, $x = (2a_0 + a_{-1})/3$ and $y = (a_0 - a_{-1})/3$.]

*

You may be interested in the article "A generalized Fibonacci sequence", by A. F. Horadam, in the American Mathematical Monthly for May 1961.

$$x \left(\frac{1 + \sqrt{5}}{2} \right)^{-1} + y \left(\frac{1 - \sqrt{5}}{2} \right)^{-1} = 1$$

$$x \left(\frac{1 + \sqrt{5}}{2} \right)^0 + y \left(\frac{1 - \sqrt{5}}{2} \right)^0 = 0$$

These are equivalent, individually, to:

$$\frac{x}{1 + \sqrt{5}} + \frac{y}{1 - \sqrt{5}} = \frac{1}{2}$$

and:

$$x + y = 0$$

Solving, we obtain the solution $\left(\frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right)$. Substitution in (*), above, leads to the desired result.

$$\begin{aligned} 2. f_4 &= \frac{(1 + \sqrt{5})^4 - (1 - \sqrt{5})^4}{2^4 \sqrt{5}} \\ &= \frac{(1 + 4\sqrt{5} + 6 \cdot 5 + 4 \cdot 5\sqrt{5} + 5^2) - (1 - 4\sqrt{5} + 6 \cdot 5 - 4 \cdot 5\sqrt{5} + 5^2)}{16\sqrt{5}} \\ &= \frac{2(4\sqrt{5} + 4 \cdot 5\sqrt{5})}{16\sqrt{5}} = \frac{8\sqrt{5}(1 + 5)}{16\sqrt{5}} = 3 \end{aligned}$$

[Before doing Exercise 3, students should plot the first few terms of the sequence a_n .]

3. $\forall_{k \geq -1} a_k = \frac{1}{3} \left(2 + \frac{(-1)^k}{2^k} \right)$ [The procedure in question leads, first, to the equation ' $2x^n - x^{n-1} - x^{n-2} = 0$ ', which reduces by factoring out ' x^{n-2} ', to ' $2x^2 - x - 1 = 0$ '. The roots of this equation are 1 and $\frac{1}{2}$. Hence, for any numbers x and y the sequence defined, for $k \geq -1$, by:

$$a_k = x \cdot 1^k + y \left(-\frac{1}{2} \right)^k \quad \left[= x + \frac{(-1)^k y}{2^k} \right]$$

The two solutions of (*) which are exponential sequences [with domain $\{j: j \geq -1\}$] are the sequences a' and a'' such that, for each $k \geq -1$,

$$a'_k = \left(\frac{1 + \sqrt{5}}{2}\right)^k \quad \text{and} \quad a''_k = \left(\frac{1 - \sqrt{5}}{2}\right)^k$$

Since $a'_0 = 1 = a''_0$, neither of these satisfies the initial conditions (**). The fact that, since a' and a'' are solutions of (*), the sequence $xa' + ya''$ is a solution, for any numbers x and y , is an important property of recursion formulas which, like (*), involve only addition and perhaps multiplication by constant sequences. For example, the recursion formula ' $a_{n+3} + 2a_{n+2} - 5a_n = 0$ ' has this property, but ' $a_{n+1} = a_n^2$ ' and ' $a_{n+1} + 2a_{n-1} = 1$ ' do not. Those which do have the property are called 'homogeneous linear recursion formulas' [the third example, above, is a nonhomogeneous linear recursion formula]. If you have studied differential equations, you may recall a discussion like that on pages 8-127 and 8-128 but dealing with the problem of solving a linear differential equation. [Instead of 'recursion formula' we might, above, have written 'difference equation'.]

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Answers for Part ★H.

1. Because of the result stated at the end of the text on page 8-128 [and because there is only one sequence which satisfies the recursive definition for f] we shall have an explicit definition for f if we can find numbers x and y such that, when, for each $k \geq 1$,

$$(*) \quad a_k = x \left(\frac{1 + \sqrt{5}}{2}\right)^k + y \left(\frac{1 - \sqrt{5}}{2}\right)^k,$$

$a_{-1} = 1$ and $a_0 = 0$. This leads to the two linear equations:

TC[8-127, 128]b

3. $m = f_k$, $n = f_{k+1}$, $p = f_{k-1}$, for $k \geq 3$ [Since, for $k \geq 1$, $f_k = f_{k-2} + f_{k-1}$, it follows that a square region of side-measure f_k (8, for $k = 6$) can be divided as shown in the first figure into two rectangular regions bounded by an upper $f_{k-2} \times f_k$ rectangle and a lower $f_{k-1} \times f_k$ rectangle. (The restriction in the answer is to ensure that $f_{k-2} \in I^+$.) The upper rectangular region can be divided, as shown, into two $f_{k-2} \times f_k$ triangular regions and the lower can (again since $f_k = f_{k-2} + f_{k-1}$) be divided into two trapezoidal regions. Since $f_{k-1} + f_k = f_{k+1}$, these regions can apparently be fitted together inside an $f_{k-1} \times f_{k+1}$ rectangle. However, since $f_{k+1}f_{k-1} - f_k^2 = (-1)^k$, the pieces will overlap by one unit of area if k is odd, and will fail to cover one unit of area if k is even. [See E. P. Northrop's Riddles in Mathematics [New York: Van Nostrand, 1944], pp. 48-50. See, also, pages 162-3 of the May 1961 issue of Scientific American and page 176 of the June 1961 issue.]

*

The roots of ' $x^2 - x - 1 = 0$ ', which are asked for on the second line from the bottom of page 8-127, are

$$\frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \frac{1 - \sqrt{5}}{2}.$$

[The opposite of the second root is the golden ratio. See Northrop, op. cit., pp. 50-54.]

[Since there is no real number x such that $x^{1-2} = 0$, there is no real number x such that, for each n , $x^{n-2} = 0$. Nevertheless, the constant sequence whose range is $\{0\}$ and whose domain is $\{j: j \geq -1\}$ does satisfy the recursion equation (*). That it doesn't arise in this analysis is due merely to the fact that we are looking at exponential sequences, and it is not one of these.]

$$(e) 3, 3\sqrt{2}, 6, 6\sqrt{2}, 12, 12\sqrt{2}, \dots$$

$$(f) \sqrt{3}, 3, 3\sqrt{3}, 9, 9\sqrt{3}, 27, \dots \quad [\sqrt{3}r^3 = 9 \iff r^3 = \sqrt{3}\sqrt{3}\sqrt{3}]$$

$$(g) \sqrt{3}, -3, 3\sqrt{3}, -9, 9\sqrt{3}, -27, \dots \quad [\sqrt{3}r^3 = -9 \iff r^3 = (-\sqrt{3})(-\sqrt{3})(-\sqrt{3})]$$

$$(h) \left. \begin{array}{l} -3, -3, -3, -3, -3, -3, \dots \\ -3, 3, -3, 3, -3, 3, \dots \end{array} \right\} \quad [-3r^4 = -3 \iff r^2 = 1]$$

$$(i) \pi, 0, 0, 0, 0, 0, \dots \quad [\pi r^4 = 0 \iff r = 0]$$

$$(j) a_1, 0, 0, 0, 0, 0, \dots \quad [\text{provided that } a_1 \neq 0]$$

$$2. (a) 2, 10, 50, 250 \quad [2r^3 = 250]$$

$$(b) \left. \begin{array}{l} 1, 4, 16, 64, 256 \\ 1, -4, 16, -64, 256 \end{array} \right\} \quad [1r^4 = 256 \iff r^2 = 16]$$

$$(c) -1, 2, -4, 8, -16, 32 \quad [-1r^5 = 32 \iff r^5 = -2^5 = (-2)^5]$$

(d) Can't be done since there is no real number 4 such that $-r^4 = 8$.

Note that in answering part (i) of Exercise 1, one has not inserted three geometric means between π and 0 because, for this GP, $a_2 = a_3 = a_4 = 0$. Similar remarks can be made with respect to both solutions for part (h). For that matter, one cannot insert even one geometric mean between say, -3 , and -3 .

Answers for the rest of Part A are in the COMMENTART for page 8-131.

[A sequence is both an AP and a GP if and only if it is a nonzero constant sequence--that is, if and only if it is a GP with a common ratio [see page 8-130] 1. [See Exercise C5 on page 8-133.] Note that there is one AP with common difference 0 which is not a GP--the sequence each of whose terms is 0.]

*

Proof of:

For any geometric progression a ,

$$r \neq 0 \Rightarrow \forall_n a_n \neq 0 \quad [\text{See line 3, page 8-130.}]$$

(i) $a_1 \neq 0$ [recursive definition of a GP]

(ii) Suppose that $a_q \neq 0$. Then [by Theorem 55], since $r \neq 0$, $a_q r \neq 0$. So, by the recursive definition of a GP, $a_{q+1} \neq 0$.

*

Notice that the general procedure for finding r is to compute the ratio a_2/a_1 . Since $a_1 \neq 0$, this ratio exists. If the common ratio is not 0, Theorem 167b can be used.

If the common ratio is 0, all the terms except the first are 0.

*

Answers for Part A.

1. (a) 1, 2, 4, 8, 16, 32, ...

(b) 3, 9, 27, 81, 243, 729, ... $[3r^3 = 81]$

(c) $\left. \begin{array}{l} -2, -1, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, \dots \\ -2, 1, -\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \end{array} \right\} [-2r^2 = -\frac{1}{2}]$

(d) -9, 3, -1, $\frac{1}{3}$, $-\frac{1}{9}$, $\frac{1}{27}$, ... $[-9r^3 = \frac{1}{3}]$

Conceivably, it may help some students to look at:

$$\begin{array}{c} \nearrow a_1, a_1 r, a_1 r^2, a_1 r^3, \dots \\ \text{[not 0]} \end{array}$$

as well as to read the recursive definition, in order to find out what a geometric progression is.

Each exponential sequence is a geometric progression whose first term is 1. Each geometric progression is the product of a nonzero constant sequence by an exponential sequence.

Especially for the discussion of the topic "infinite geometric progressions" and for students' later transition to the study of other "power series", it is desirable to include all exponential sequences, among geometric progressions [on this point, recall the discussion referring to Theorem 168b in the COMMENTARY for page 8-101]. In particular, the exponential sequence with base 0, for which $a_1 = 1$ and, for each $n \geq 1$, $a_n = 0$, should be considered a geometric progression. However, there seems to be no adequate reason for including the constant sequence each of whose terms is 0. And, excluding it by the requirement ' $a_1 \neq 0$ ', ensures that each geometric progression has a unique common ratio, as this is defined at the top of page 8-130.

*

- | | | | |
|-----------------------|---|-------------|----------------------------|
| (1) $r = 2$ | (2) not a GP [$4 = 2 \cdot 2$ but $6 \neq 4 \cdot 2$] | | |
| (3) $r = \frac{1}{2}$ | (4) $r = -3$ | (5) $r = 0$ | (6) not a GP [$a_1 = 0$] |
| (7) $r = \frac{3}{2}$ | (8) not a GP [$a_1 = 0$] | (9) $r = 1$ | |

The sequences (2), (8), and (9) could be APs.

- | | | |
|-------------|-------------|-------------|
| (2) $d = 2$ | (8) $d = 0$ | (9) $d = 0$ |
|-------------|-------------|-------------|

3. If, for $1 + a > 0$, one approximates $\sqrt{1+a}$ by the dividing-and-averaging method, taking 1 as his first approximation, y_1 , then

$$y_2 = \frac{1 + \frac{1+a}{1}}{2} = \frac{2+a}{2} = 1 + \frac{a}{2},$$

and, by (*) of Exercise 1 and (c) and (d) of Exercise 2, $(1 + \frac{a}{2})$

- $\sqrt{1+a}$ is between $\frac{[1^2 - (1+a)]^2}{8(1+a)1}$ and $\frac{[1^2 - (1+a)]^2}{8 \cdot 1^3}$. So, the error

in y_2 is between $\frac{a^2}{8(1+a)}$ and $\frac{a^2}{8}$. If a is near 0 then a^2 is very

small. For example, with $a = 0.02$, one finds that $\sqrt{1.02}$ is approximately 1.01, the error being between 0.00005 and 0.000049. So,

$$1.00995 < \sqrt{1.02} < 1.009951.$$

[If one takes an additional dividing-and-averaging step, he obtains, as a third approximation to $\sqrt{1+a}$,

$$1 + \frac{a}{2} - \frac{a^2}{8} + \frac{a^3}{16(1 + \frac{a}{2})}$$

This suggests taking ' $1 + a/2 - a^2/8$ ' as a more convenient expression for obtaining good approximations for square roots of numbers near 1. Using (*), (c), and (d), again, one finds that

$$\sqrt{1+a} - (1 + \frac{a}{2} - \frac{a^2}{8})$$

is between

$$\frac{8a^3(1 + \frac{a}{2})^2 - a^4}{128(1 + \frac{a}{2})^3} \quad \text{and} \quad \frac{8a^3(1+a) - a^4}{128(1+a)(1 + \frac{a}{2})}$$

For $a = 0.02$, this shows that

$$1.0099504938361483 < \sqrt{1.02} < 1.0099504938362672.]$$

etc.]. Suppose, now, that $a > b$. It follows, since $\sqrt{a} > 0$, that $\sqrt{a}\sqrt{a} > \sqrt{a}\sqrt{b}$ and, so, that $a + 2\sqrt{a}\sqrt{b} + b < a + 2\sqrt{a}\sqrt{a} + a = 4a$. Hence, $(\sqrt{a} + \sqrt{b})^2 < 4a$. On the other hand, since it follows that $\sqrt{a} > \sqrt{b}$, $\sqrt{a} \neq \sqrt{b}$ and, by Theorem 97b, $a + b > 2\sqrt{a}\sqrt{b}$. So, $a + 2\sqrt{a}\sqrt{b} + b > 4\sqrt{a}\sqrt{b} > 4\sqrt{b}\sqrt{b} = 4b$. Hence, $(\sqrt{a} + \sqrt{b})^2 > 4b$. Consequently, for $a > 0$ and $b > 0$, if $a > b$ then $4b < (\sqrt{a} + \sqrt{b})^2 < 4a$.

(b) For $a > 0$ and $b > 0$, $a - b = (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$ and, so,

$$(\sqrt{a} - \sqrt{b})^2 = \frac{(a - b)^2}{(\sqrt{a} + \sqrt{b})^2}. \text{ Suppose, now that } a > b. \text{ It follows,}$$

by Theorems 87 and 97b, that $(a - b)^2 > 0$ and, by the theorem proved in part (a), above, $4a > (\sqrt{a} + \sqrt{b})^2$ and $(\sqrt{a} + \sqrt{b})^2 > 4b$.

Since $\sqrt{a} + \sqrt{b} > 0$, it follows, by Theorem 100, that

$$\frac{(a - b)^2}{(\sqrt{a} + \sqrt{b})^2} > \frac{(a - b)^2}{4a}. \text{ Since } b > 0, \text{ it follows, again by}$$

Theorem 100, that $\frac{(a - b)^2}{4b} > \frac{(a - b)^2}{(\sqrt{a} + \sqrt{b})^2}$. Hence, if $a > b$ then

$$\frac{(a - b)^2}{4a} < (\sqrt{a} - \sqrt{b})^2 < \frac{(a - b)^2}{4b}.$$

Consequently, the theorem.

(c) For $y_1 > 0$, $y_1^2 > 0$ and $\sqrt{y_1^2} = y_1$. So, by the theorem of part

(b), for $y_1 > 0$ and $a > 0$,

$$\text{if } y_1^2 > a \text{ then } \frac{(y_1^2 - a)^2}{8y_1^2} < \frac{(y_1 - \sqrt{a})^2}{2} < \frac{(y_1^2 - a)^2}{8a}.$$

The desired result follows on dividing by y_1 .

(d) As in part (c), for $a > 0$ and $y_1 > 0$,

$$\text{if } y_1^2 > a \text{ then } \frac{(a - y_1^2)^2}{8a} < \frac{(\sqrt{a} - y_1)^2}{2} < \frac{(a - y_1^2)^2}{8y_1}.$$

Mathematics, January 1961, pp. 45 and 46. The harmonic mean of two positive numbers is defined in Exercise 25 on page 8-222 [see, also, Exercise 9, on page 8-131]. If, in the figure on page 8-131, E is the foot of the perpendicular from C to \overline{BP} then $EP = 2ab/(a + b)$, the harmonic mean of a and b. Arguments similar to those given in the answer, above, for Exercise 8 show that $EP < PC$. So, the harmonic mean of two positive numbers is less than their geometric mean.]

9. the geometric mean of x and y [or: \sqrt{xy}] [$\frac{x+y}{2} \cdot \frac{2xy}{x+y} = xy$]
*

Answers for Part ★B.

1. For $a > 0$ and $b > 0$, $a = (\sqrt{a})^2$, $b = (\sqrt{b})^2$, and $\sqrt{ab} = \sqrt{a}\sqrt{b}$. Hence,

$$\frac{a+b}{2} - \sqrt{ab} = \frac{(\sqrt{a})^2 + (\sqrt{b})^2 - 2\sqrt{a}\sqrt{b}}{2} = \frac{(\sqrt{a} - \sqrt{b})^2}{2}$$

$$\text{Consequently, } \forall_{x>0} \forall_{y>0} \frac{x+y}{2} - \sqrt{xy} = \frac{(\sqrt{x} - \sqrt{y})^2}{2}.$$

From the algebra theorem just proved it follows [for $y_1 > 0$ and $a > 0$] that

$$\begin{aligned} \frac{y_1 + \frac{a}{y_1}}{2} - \sqrt{y_1 \cdot \frac{a}{y_1}} &= \frac{(\sqrt{y_1} - \sqrt{a/y_1})^2}{2} \\ &= \frac{(\sqrt{y_1} - \sqrt{a/y_1})^2}{2} \cdot \frac{(\sqrt{y_1})^2}{(\sqrt{y_1})^2} \\ &= \frac{[(\sqrt{y_1} - \sqrt{a/y_1}) \sqrt{y_1}]^2}{2y_1} = \frac{(y_1 - \sqrt{a})^2}{2y_1} \end{aligned}$$

2. (a) For $a > 0$ and $b > 0$, $(\sqrt{a} + \sqrt{b})^2 = a + 2\sqrt{a}\sqrt{b} + b$, and, if $a > b$ then, by Theorem 98b, $\sqrt{a} > \sqrt{b}$ [recall that, for $a > 0$, $(\sqrt{a})^2 = a$,

prove the theorem given in Exercise 7. Exercise 5 illustrates the fact that the arithmetic mean of two numbers is greater than their geometric mean:

$$\text{if } a \neq b \text{ then } \frac{a+b}{2} > \sqrt{ab}$$

Reviewing previous theorems on inequations, this looks suspiciously like Theorem 97b.]

7. For $a > 0$ and $b > 0$, \sqrt{a} and \sqrt{b} are positive numbers such that $(\sqrt{a})^2 = a$ and $(\sqrt{b})^2 = b$. Suppose that $a = b$. It follows that $\sqrt{a} \neq \sqrt{b}$ [if $\sqrt{a} = \sqrt{b}$ then $(\sqrt{a})^2 = (\sqrt{b})^2$, and $a = b$]. Hence, by Theorem 97b, $(\sqrt{a})^2 + (\sqrt{b})^2 > 2\sqrt{a}\sqrt{b}$. Since $\sqrt{a}\sqrt{b} = \sqrt{ab}$ [\sqrt{ab} is, by definition the positive number whose square is ab , while $\sqrt{a}\sqrt{b}$ is positive and $(\sqrt{a}\sqrt{b})^2 = (\sqrt{a})^2(\sqrt{b})^2 = ab$], it follows that $a + b > 2\sqrt{ab}$. Hence [since $2 > 0$], for $a > 0$ and $b > 0$,

$$\text{if } a \neq b \text{ then } \frac{a+b}{2} > \sqrt{ab}$$

Consequently, $\forall_{x>0} \forall_{y>0} [x \neq y \Rightarrow \frac{x+y}{2} > \sqrt{xy}]$.

8. Given two positive numbers, a and b , let A and D be points such that $AD = a + b$, let C be the point of \overline{AD} such that $AC = a$ [and $CD = b$], and let B be the midpoint of \overline{AD} . Since $a \neq b$, $C \neq B$. A ray with vertex C and perpendicular to \overleftrightarrow{AD} intersects the circle with center B and radius BD at a point P . Since \overline{BP} and \overline{PC} are hypotenuse and leg, respectively, of right $\triangle BPC$, it follows that $BP > PC$. Since the diameter of the circle is $a + b$, $BP = (a+b)/2$. Since $\overline{PC} \perp \overline{AD}$, PC is a mean proportional between \overline{AC} and \overline{CD} and, since $AC = a$ and $CD = b$, it follows that $PC = \sqrt{ab}$. Hence, $(a+b)/2 > \sqrt{ab}$. [This proof is given in the article "A Geometric Construction for the Arithmetic Mean, the Geometric Mean and the Harmonic Mean of Two Positive Numbers", by Adrian L. Hess, School Science and

3. When the product of the two numbers is positive. [This answers both questions.] [As remarked in connection with Exercise 2 of Part A, one can insert one or more geometric means only between two numbers, neither of which is 0. In order to be able to insert an odd number $2n - 1$ of geometric means between two nonzero numbers x and y , it is necessary and sufficient that there be a real number r such that $r^{2n} = x/y$. Such a number r will be different from 0 and, by Theorems 157, 97a and 151a, will exist only if $x/y > 0$. Since, as students will learn in Unit 9, each positive number has, for each n , exactly two $(2n)$ th roots, there are two such numbers r if $x/y > 0$. Since $y \neq 0$ and, for $y \neq 0$, $x/y > 0$ if and only if $xy > 0$, an odd number of geometric means can be inserted between two numbers if and only if the product of the two numbers is positive.]

*

Although one can insert one geometric mean between x and y if and only if $x \neq y$ and $xy > 0$, x and y have a geometric mean even if $x = y$. As in Exercise 4, the geometric mean of x and y is \sqrt{xy} and, for simplicity, is defined only for $x > 0$ and $y > 0$.

*

4. (a) $2\sqrt{2}$

(b) 3

(c) Suppose that a , b , and c are positive numbers which are consecutive terms of a GP. By definition, there is a number r such that $b = ar$ and $c = br$. From this it follows that $ac = a(br) = b^2$ and, since $ac > 0$ and $b > 0$, that $b = \sqrt{ac}$.

5. the arithmetic mean [assuming that you prefer to pass]
 $[(40 + 90)/2 = 65, \sqrt{40 \cdot 90} = 60]$

6. Theorem 97b [This question is designed to lead students to state and

TC[8-131, 132]a

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is a d and there is an r such that $\forall_n a_1 + (n - 1)d = a_1 r^{n-1}$. In particular, $a_1 + (2 - 1)d = a_1 r^{2-1}$ and $a_1 + (3 - 1)d = a_1 r^{3-1}$. So, $a_1(r - 1)^2 = 0$. That is, $a_1 = 0$ or $r = 1$. But, since a is a GP, $a_1 \neq 0$. So, $r = 1$. Also, for each n , $a_{n+1} = a_n \cdot 1 = a_n$. Therefore, a is a constant sequence, and since $a_1 \neq 0$, it is a nonzero constant sequence.

*

Answers for Part D.

1. 8

2. $341/512$

3. $-1, 0$

Answers to the rest of Part D are in the COMMENTARY for page 8-135.

Answers for Part C.

1. (i) $a_1 = a_1 1 = a_1 r^{1-1}$

(ii) $a_{q+1} = a_q r = a_1 r^{q-1} r = a_1 r^{q+1-1}$

$\left[\begin{array}{c} \text{definition} \\ \text{of GP} \end{array} \right]$
 $\left[\begin{array}{c} \text{inductive} \\ \text{hypothesis} \end{array} \right]$

2. By Theorem 167a, $s_n = \sum_{p=1}^n a_1 r^{p-1}$. By Theorem 153 [and Theorem

133], $(r-1) \sum_{p=1}^n a_1 r^{p-1} = a_1 (r^n - 1)$. So, [for $r \neq 1$], $\sum_{p=1}^n a_1 r^{p-1}$

$= \frac{a_1 (r^n - 1)}{r - 1} = \frac{a_1 (1 - r^n)}{1 - r}$. Hence, $s_n = \frac{a_1 (1 - r^n)}{1 - r}$.

3. $s_n = \frac{a_1 (1 - r^n)}{1 - r} = \frac{a_1 - a_1 r^n}{1 - r} = \frac{a_1 - a_1 r^{n-1} \cdot r}{1 - r} = \frac{a_1 - a_n r}{1 - r}$

\uparrow Theorem 167c \uparrow Theorem 167a

4. na_1 [If $r = 1$ then a is a constant sequence. Use Theorems 133 and 132a.]

5. Suppose that a is a nonzero constant sequence. Then, $\forall_n a_{n+1} = a_n \neq 0$. Since $\forall_n a_{n+1} - a_n = 0$, it follows that Δa is a constant. Hence, a is an AP. Also, $a_1 = 0$ and $\forall_n a_{n+1} = a_n \cdot 1$. Hence, a is a GP.

On the other hand, suppose that a is an AP and a GP. Then, there

(d) From the work in part (c) we note that

$$\frac{3}{11} - s_m = \frac{3}{11 \cdot 10^{2m}}.$$

So, we wish to find the smallest m such that

$$\frac{3}{11 \cdot 10^{2m}} < 10^{-100}.$$

Since, by Theorem 152a, 10^{2m} is positive, the last inequation is equivalent to:

$$\frac{3}{11} < 10^{2m - 100}$$

and to:

$$100^{m - 50} > \frac{3}{11}$$

Now, if $m = 50$, $100^{m - 50} = 100^0 = 1 > \frac{3}{11}$. So, 50 is a number which satisfies the given inequation. But, is it the smallest? If $m = 49$, $100^{m - 50} = 0.01 < \frac{3}{11}$. In view of Theorem 152c and the transitivity of $<$, 50 is the smallest positive integer which works.

(e) This generalization expresses what students should have been noticing--the more terms used in computing the continued sum, the closer it gets to $\frac{3}{11}$.

$$\frac{3}{11} - s_n < \frac{3}{11} - s_m$$

$$\iff \frac{3}{11} \cdot \frac{1}{10^{2n}} < \frac{3}{11} \cdot \frac{1}{10^{2m}}$$

$$\iff 10^{2n} > 10^{2m}$$

$$\iff 100^{n - m} > 1$$

Now [for $n > m$], $n - m > 0$. So, by Theorems 151a and 104, $100^{n - m} \geq 1$. If $100^{n - m} = 1$, it follows, using Theorem 161, that $n - m = 0$. But $n - m > 0$. So, $100^{n - m} > 1$. Consequently, for $n > m$, $100^{n - m} > 1$. So, for $n > m$, $\frac{3}{11} - s_n < \frac{3}{11} - s_m$.

$$m_1 = m_2 = \frac{1}{2} m_0$$

Let us now consider the case where the two particles are moving in opposite directions with equal speeds v relative to the center of mass frame S . In this case, the total momentum is zero, and the total energy is $2\gamma m_0 c^2$.

$$E_{\text{total}} = 2\gamma m_0 c^2 = 2m_0 c^2 \gamma$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$E_{\text{total}} = 2m_0 c^2 \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$E_{\text{total}} = 2m_0 c^2 \gamma$$

Now, let us consider the case where the two particles are moving in the same direction with equal speeds v relative to the center of mass frame S . In this case, the total momentum is not zero, and the total energy is $2\gamma m_0 c^2$. However, the total momentum is $2\gamma m_0 v$. This means that the center of mass is moving with velocity v relative to the frame S . In this case, the total energy is $2\gamma m_0 c^2$, and the total momentum is $2\gamma m_0 v$. The total energy and total momentum are related by the equation $E_{\text{total}}^2 - (pc)^2 = (m_0 c^2)^2$, where p is the total momentum.

$$E_{\text{total}}^2 - (pc)^2 = (m_0 c^2)^2$$

$$E_{\text{total}} = \gamma m_0 c^2$$

$$E_{\text{total}} = \gamma m_0 c^2$$

$$E_{\text{total}} = \gamma m_0 c^2$$

$$E_{\text{total}} = \gamma m_0 c^2$$

$$s_n = \frac{3}{11} \left(1 - \frac{1}{10^{2n}} \right)$$

Then, $s_3 = \frac{3}{11} (0.999999) = 3 \cdot 0.090909 = 0.272727$. Etc.

The second procedure is probably more helpful in doing part (b).

(b) The simplest procedure is the following:

$$s_n = \frac{3}{11} \left[1 - \left(\frac{1}{10} \right)^{2n} \right]$$

By Theorem 152a, $\left(\frac{1}{10} \right)^{2n} > 0$. So, $1 - \left(\frac{1}{10} \right)^{2n} < 1$. Hence,

$$\frac{3}{11} \left[1 - \left(\frac{1}{10} \right)^{2n} \right] < \frac{3}{11}.$$

A student who used the first procedure in solving part (a) might use intuition in noticing that the decimal fraction equivalent for $\frac{3}{11}$ is the repeating decimal $'.272727\dots'$ and that the sum s_n is represented by a decimal fraction obtained by truncating the repeating decimal just beyond the n th '7'. But, this is not a proof. In particular, the idea that a repeating decimal is indeed a numeral is the burden of a lengthy development starting on page 8-138. Some of the reasoning used in the above proof will help in foreshadowing the work on pages 8-138ff.

$$(c) \frac{3}{11} - s_3 = \frac{3}{11} - \frac{3}{11} \left(1 - \frac{1}{10^6} \right) = \frac{3}{11} \cdot \frac{1}{10^6};$$

$$\frac{3}{11} - s_4 = \frac{3}{11} \cdot \frac{1}{10^8}; \quad \frac{3}{11} - s_5 = \frac{3}{11} \cdot \frac{1}{10^{10}}$$

$$[\text{Alternatively, } \frac{3}{11} - s_3 = \frac{3}{11} - .272727$$

$$= \frac{3}{11} - \frac{272727}{10^6}$$

$$= \frac{3 \cdot 10^6 - 2999997}{11 \cdot 10^6} = \frac{3}{11 \cdot 10^6}. \quad \text{Etc.}]$$

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4. 64 $[a_1 \cdot 2^{-7} = 2^{-1}]$ 5. 177147 $[3\sqrt{3}(\sqrt{3})^{19} = 3(\sqrt{3})^{20} = 3 \cdot 3^{10} = 3^{11}]$

6. 2 $[a_1 r^2 = 2^{-3} \text{ and } a_1 r^5 = 2^{-9}; \text{ so, } 2^{-3} r^3 = 2^{-9}. \text{ Hence, } r = 2^{-2}$
and $a_1 = 2^1.]$

7. 1111.111111 [If students use a formula to solve this problem, ask them to tell you the sum of, say, the first 19 terms.]

8. 7174453/1594323 $[a_1 = 3, r = 3^{-1}]$

9. $\frac{t - s^{20} t^{11}}{1 - s^2 t}$ [Students may be surprised at what happens if they try to simplify this fraction by using the division-with-remainder algorithm.]

10. two [19683 and 59049] 11. two $[1/19683 \text{ and } 1/59049]$
[In solving Exercise 10 students may notice that, since $3^2 < 10 < 3^3$, there can be at most three terms of the progression between any two successive powers of 10. In solving Exercise 11, they should realize that ' $10^{-5} < 3^{-(n-1)} < 10^{-4}$ ' is equivalent to ' $10^4 < 3^{n-1} < 10^5$ ' and, so, that the answer for Exercise 11 is the same as that for Exercise 10.]

12. There is none at all, let alone a smallest.

$$\left[\frac{1 - 3^{-n}}{1 - 3^{-1}} = \frac{3}{2} (1 - 3^{-n}) < \frac{3}{2} \text{ for each } n. \right]$$

13. (a) There are two ways a student might proceed here. He might notice that the terms of the progression are

$$0.27, 0.0027, 0.000027, \dots$$

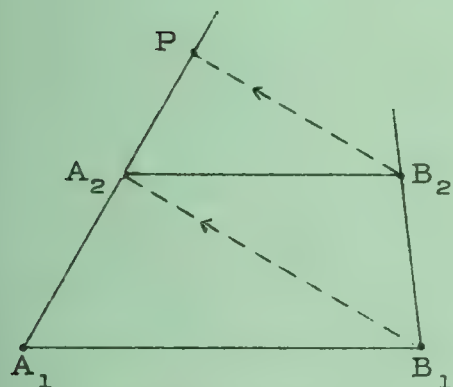
So, obviously, $s_3 = 0.272727$, $s_4 = 0.27272727$, and $s_5 = 0.2727272727$. Or, he might use the formula for s_n and get:

Answers for Part E.

- It is instructive to draw the diagram using the recursive formulas as instructions. Start with $\triangle A_1OB_1$. Step 1 consists in picking any point A_2 of $\overline{A_1O}$ and drawing the line through A_2 parallel to $\overline{A_1B_1}$. Let B_2 be the point in which this parallel intersects $\overline{OB_1}$. Step 2 consists of drawing the line through B_2 parallel to $\overline{B_1A_2}$. Let A_3 be the point in which this parallel intersects $\overline{OA_1}$. Etc.

An interesting observation is that no matter where on $\overline{A_1O}$ the point A_2 is selected ["even if it is real close to O"], the line through B_2 [parallel to $\overline{B_1A_2}$] which intersects the half-line $\overrightarrow{A_2O}$ does so in a

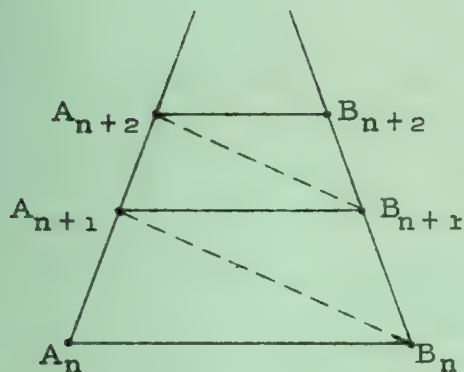
point between A_2 and O. [The justification for this is interesting and instructive. It will help students get started on part (a), and it is essential for part (c).] For, suppose that P is this point of intersection. By the angle-angle similarity theorem, $A_2B_2P \longleftrightarrow A_1B_1A_2$ is a similarity. So, there is an $r > 0$ such that $A_2B_2 = r A_1B_1$ and $A_2P = r A_1B_2$. Also, $OA_2B_2 \longleftrightarrow OA_1B_1$ is a simi-



larity. Since $A_2B_2 = r A_1B_1$, it follows that $A_2O = r A_1O$. Now, $A_1A_2 < A_1O$. So, $r A_1A_2 < r A_1O$. That is, $A_2P < A_2O$. Hence, since $P \in \overrightarrow{A_2O}$, $P \in \overline{A_2O}$.

(continued on TC[8-136]b)

(a)



$OA_{n+1} B_{n+1} \leftrightarrow OA_n B_n$ is a simi-

larity; so, $\frac{A_{n+1} B_{n+1}}{A_n B_n} = \frac{OA_{n+1}}{OA_n}$.

Also, $A_{n+2} A_{n+1} B_{n+1} \leftrightarrow A_{n+1} A_n B_n$

is a similarity; so, $\frac{A_{n+1} B_{n+1}}{A_n B_n}$

$= \frac{OA_{n+1} - OA_{n+2}}{OA_n - OA_{n+1}}$. Therefore,

$\frac{OA_{n+1}}{OA_n} = \frac{OA_{n+1} - OA_{n+2}}{OA_n - OA_{n+1}}$. From

this it follows that $\frac{OA_{n+1}}{OA_n} = \frac{OA_{n+2}}{OA_{n+1}}$. But, since $OA_{n+2} B_{n+2}$

$\leftrightarrow OA_{n+1} B_{n+1}$ is a similarity, $\frac{A_{n+2} B_{n+2}}{A_{n+1} B_{n+1}} = \frac{OA_{n+2}}{OA_{n+1}}$. Thus,

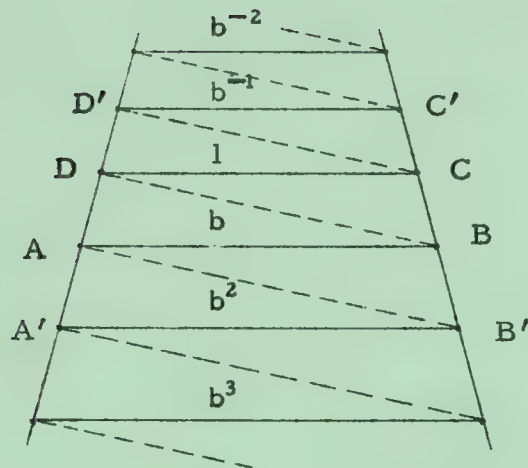
$$\frac{A_{n+1} B_{n+1}}{A_n B_n} = \frac{A_{n+2} B_{n+2}}{A_{n+1} B_{n+1}}.$$

So, since $A_1 B_1 \neq 0$, the sequence $A_1 B_1, A_2 B_2, A_3 B_3, \dots$ is a geometric progression.

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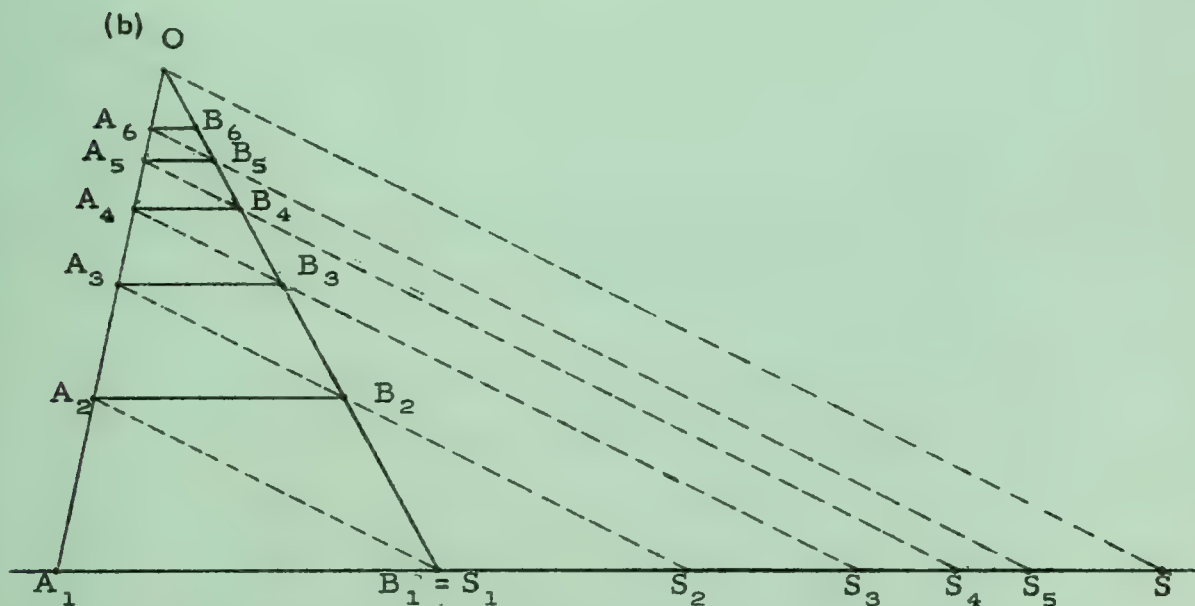
Part (a) suggests a geometric construction of the terms of a two-way exponential sequence with base b [See Exercise 5 on page 8-105]. The construction involves drawing a trapezoid whose bases are segments of measures b and 1 , respectively, extending the legs of the trapezoid, drawing one of the diagonals, and then constructing parallels to the bases and parallels to the diagonal in a zig-zag fashion.

[See figure on following page.]



There are other geometric progressions which arise as by-products of this construction. The measures of the diagonals [dashed segments] are in geometric progression. So are the measures of the collinear sides of the trapezoids.

*



The key to this exercise is to notice that quadrilateral $S_n A_{n+1} B_{n+1} S_{n+1}$ is a parallelogram. The theorem is easily

established by induction ["easily" because you are expected to ignore the Introduction-to-Unit-6 matters].

$$(i) \quad \sum_{p=1}^1 A_p B_p = A_1 B_1; B_1 = S_1$$

$$(ii) \quad \left. \begin{aligned} \sum_{p=1}^{q+1} A_p B_p &= \sum_{p=1}^q A_p B_p + A_{q+1} B_{q+1} \\ &= A_1 S_q + A_{q+1} B_{q+1} \\ &= A_1 S_q + S_q S_{q+1} \\ &= A_1 S_{q+1} \end{aligned} \right\} \begin{array}{l} \text{[inductive hypothesis]} \\ \text{[opposite sides of a} \\ \text{parallelogram are} \\ \text{congruent]} \end{array}$$

[Students should see that $A_1 S_n$ is the sum of the first n terms of the geometric progression mentioned in part (a).]

- (c) As mentioned in the discussion preceding the answer to part (a), $A_{n+1} \in \overline{A_1 O}$. Also, $\overline{A_{n+1} S_n} \parallel \overline{OS}$. So, from geometry, $S_n \in A_1 S$. Hence, $A_1 S_n < A_1 S$. [Students should conclude from this part that the range of the continued sum sequence for the geometric progression of part (a) has an upper bound but not a greatest member.]

2. (a) The problem is full of isosceles right triangles. The sequence of area-measures, starting with that of $\triangle P_0 P_1 P_2$, is the geometric progression

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$$

$$\text{So, } \forall_n \sum_{k=1}^n K(\triangle P_{k-1} P_k P_{k+1}) = \sum_{k=1}^n \left(\frac{1}{2}\right)^k = 1 - \left(\frac{1}{2}\right)^n.$$

(b) Intuitively, students should see that no matter how many such triangles we form, the sum of their area-measures is less than the area-measure of the big triangle. [The point P_n is between P_1 and A or between P_0 and A. This follows from the geometric considerations discussed in the answers to Exercise 1.] So, it seems to be the case that the smallest x is 1. Now, let's prove this. From part (a) we know that we are seeking the smallest x such that

$$(*) \quad \forall_n \quad 1 - \left(\frac{1}{2}\right)^n < x.$$

Since $1/2 > 0$, it follows from Theorem 152a that, for each n , $(1/2)^n > 0$ and, so, that $1 - (1/2)^n < 1$. Hence, 1 satisfies (*). On the other hand, if $c < 1$ then $1 - c > 0$ and, by Theorem 165, since $0 < 1/2 < 1$, it follows that $(1/2)^n < 1 - c$ for all sufficiently large n . Hence, $1 - (1/2)^n > c$ for some n , and so, it is not the case that $1 - (1/2)^n < c$ for each n . Consequently, c does not satisfy (*). Since 1 satisfies (*) and no number $c < 1$ satisfies (*), 1 is the smallest number which satisfies (*).

The rest of Part E is treated in the COMMENTARY for page 8-137.

(b) Since $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, we have $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

On the other hand, since $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, we have $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

By the definition of $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, we have $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$.
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 We have that $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

(b) Since, for each n , $\frac{1}{n+1} > 0$, it follows from the result of part (a) that

$$\forall_n \sum_{p=1}^n P_p P_{p+1} < 1.$$

On the other hand, suppose that $c < 1$. Then, $1 - c > 0$ and

$$\frac{1}{n+1} < 1 - c \text{ if and only if } n > \frac{1}{1-c} - 1.$$

By the cofinality principle, there is a positive integer n such that $n > \frac{1}{1-c} - 1$ and, hence, such that $\frac{1}{n+1} < 1 - c$ --that is, such that $1 - \frac{1}{n+1} > c$. Consequently, it is not the case that $1 - \frac{1}{n+1} < c$ for each n . Hence, 1 is the smallest number x

such that $\forall_n \sum_{p=1}^n P_p P_{p+1} < x$.

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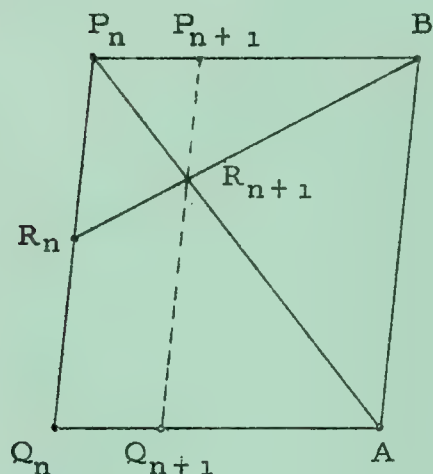
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We begin by finding a recursive definition for the sequence a



such that, for each n , $a_n = P_n B$. For each n , let Q_n be the point at which $\overrightarrow{P_n R_n}$ and $\overrightarrow{R_{n+1} A}$ intersect, and let

$$r_n = \frac{R_n Q_n}{P_n R_n}.$$

Since $AR_{n+1}B \longleftrightarrow P_n R_{n+1}R_n$ is a similarity,

$$\frac{AB}{P_n R_n} = \frac{R_{n+1} A}{P_n R_{n+1}}.$$

Since $AR_{n+1}Q_{n+1} \longleftrightarrow P_n R_{n+1}P_{n+1}$ is a similarity,

$$\frac{AR_{n+1}}{P_n R_{n+1}} = \frac{R_{n+1} Q_{n+1}}{R_{n+1} P_{n+1}} = \frac{Q_{n+1} A}{P_{n+1} P_n}.$$

Now, $AB = P_n R_n + R_n Q_n$; so, $AB/P_n R_n = 1 + r_n$. Also, $Q_{n+1} A = P_{n+1} B = a_{n+1}$ and $P_{n+1} P_n = a_n - a_{n+1}$. Consequently,

$$1 + r_n = \frac{R_{n+1} A}{P_n R_{n+1}} = r_{n+1} = \frac{a_{n+1}}{a_n - a_{n+1}}.$$

Since $Q_1 = R_1$, it follows that $r_1 = 0$ and since, for each n , $r_{n+1} = r_n + 1$, it is easy to see [and to prove by induction] that, for each n , $r_n = n - 1$. Since $r_{n+1} (a_n - a_{n+1}) = a_{n+1}$, it follows that, for each n ,

$$a_{n+1} = \frac{r_{n+1}}{r_{n+1} + 1} a_n = \frac{n}{n+1} a_n.$$

Since $a_1 = P_1 B = R_1 A = 1$, it is easy to see [and to prove by induction] that, for each n , $a_n = 1/n$. Consequently,

$$\forall_n \sum_{p=1}^n P_p P_{p+1} = 1 - \frac{1}{n+1}.$$

3. (a) A key geometry idea here is that a median of a triangle bisects the region bounded by the triangle. A second important geometry idea is that the line parallel to one side of a triangle and containing the midpoint of a second side bisects the third. [This second notion allows us to conclude that $\overline{A_{n+1}M_{n+1}}$ is the median from A_{n+1} of $\triangle M_n A_{n+1} C$.] Thus, the sequence of area-measures of the shaded regions is the geometric progression

$$\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \dots$$

$$\text{So, } \sum_{p=1}^n K(\triangle A_p M_p A_{p+1}) = \sum_{p=1}^n \left(\frac{1}{4}\right)^p = \frac{1}{4} \left(\frac{1 - \left(\frac{1}{4}\right)^{n+1}}{1 - \frac{1}{4}} \right) = \frac{1}{3} \left(1 - \left(\frac{1}{4}\right)^{n+1} \right).$$

- (b) For each of the trapezoids, the longer base is twice as long as the shorter one. So, using the formulas for the area-measures of triangular and trapezoidal regions it is easy to show $\left[\frac{1}{2}h(b+2b) = 3 \cdot \frac{1}{2}hb\right]$ that each trapezoidal region is 3 times the corresponding [shaded] triangular region. Hence, [using the result of part (a)],

$$\sum_{p=1}^n K(\triangle A_p B_p B_{p+1} A_{p+1}) = 1 - \left(\frac{1}{4}\right)^{n+1}.$$

- (c) The smallest x is 1. [See answer to Exercise 2(b).]

$$(d) \frac{1}{3} \left[\frac{1}{3} \left(1 - \left(\frac{1}{4}\right)^{n+1} \right) \right] < x \iff 1 - \left(\frac{1}{4}\right)^{n+1} < 3x$$

$$\star 4. (a) \cdot \sum_{p=1}^n P_p P_{p+1} = P_1 P_{n+1} \text{ and, since } P_1 B = R_1 A = 1, P_1 P_{n+1} = 1 - P_{n+1} B.$$

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This example should correct any incorrect generalization made by students to the effect that the terms of a sequence--say, a continued sum sequence--necessarily increase toward the limit of the sequence. A repeating decimal stands for the sum of an infinite geometric progression the terms of whose continued sum sequence are increasing. [This is because each term of the geometric progression is positive.] So, from working with repeating decimals one may get the impression that the terms of a continued sum sequence "creep up toward the limit", and are all less than the limit. This is the case for the continued sum sequences of a geometric progression a such that $a_1 > 0$ and $0 < r < 1$ [if $a_1 < 0$ and $0 < r < 1$, the terms of the continued sum sequence "creep down toward the limit"]. However, if $-1 < r < 0$, the terms of the continued sum sequence oscillate about the limit, and if $r = 0$ the continued sum sequence is a constant whose limit is a_1 . If $r \geq 1$, the continued sum sequence has no limit, since its terms either increase above any bound [if $a_1 > 0$] or decrease below any bound [if $a_1 < 0$]. If $r = -1$, the terms of the continued sum sequence are alternately a_1 and 0 and, since $a_1 \neq 0$, do not cluster about any number. If $r < -1$, the terms of the continued sum sequence oscillate more and more wildly and do not approach any limit. [The preceding remarks summarize the discussion on page 8-143 and Theorem 168 on page 8-144.]

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from the handwriting of the person who

it is a matter of fact that the handwriting is not the same as the handwriting of the person who wrote the letter of the 10th of 1887.

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by reading the quantifiers properly:

for each $y > 0$, there is an m such that, for each $n \geq m$,

This last assertion says that the inequation is satisfied by all sufficiently large values of 'n', but gives no information as to how large these values must be. The preceding assertion tells you that a value of 'n' is sufficiently large if it is not less than $10/(9y)$.

Notice that, in passing from the first assertion to the second, one uses the fact that $\forall_{y>0} \exists_m m \geq 10/(9y)$ --a consequence of the cofinality principle--and the transitivity of \geq . [Instead of the cofinality principle one can use the fact that $\lceil 10/(9y) \rceil + 1$ is a (positive) integer greater than $10/(9y)$ --but, of course, the cofinality principle is used to establish the existence of the greatest integer function.]

*

The word 'infinite' in 'infinite geometric progression' is superfluous. By definition, each geometric progression has infinitely many terms. It would be sufficient to speak of the sum of all the terms of the GP, were it not for the fact that students need to become acquainted with the customary phrase used in lines 4 and 5 on page 8-140.

*

lines 1 - 5 from bottom of page 8-140: The 7th term is $\frac{43}{64}$, the 8th term: $\frac{85}{128}$ and the 9th term is $\frac{171}{256}$.

Since $\sum_{p=1}^n \left(-\frac{1}{2}\right)^{p-1} = \frac{1 - \left(-\frac{1}{2}\right)^n}{1 - -\frac{1}{2}} = \frac{2}{3} \left(1 - \left(-\frac{1}{2}\right)^n\right)$, the 100th term is

$\frac{2}{3} \left(1 - \left(\frac{1}{2}\right)^{100}\right)$ and the 101st term is $\frac{2}{3} \left(1 + \left(\frac{1}{2}\right)^{101}\right)$. It is pretty easy

to guess that $\sum_{p=1}^{\infty} \left(-\frac{1}{2}\right)^{p-1}$ is $\frac{2}{3}$. The terms of the continued sum

sequence oscillate around $\frac{2}{3}$, getting closer to $\frac{2}{3}$ with each oscillation.

It is a well known fact that the human mind is capable of performing a wide range of tasks, from simple arithmetic to complex logical reasoning. This ability is often referred to as the "power of the mind".

The power of the mind is a result of the complex structure of the human brain, which is capable of processing information in a highly efficient manner. This is why we are able to perform tasks that are often considered to be beyond the capabilities of machines.

One of the most remarkable aspects of the human mind is its ability to learn from experience. This is a process that involves the brain's ability to store information and retrieve it when needed. This is why we are able to perform tasks that we have never done before, as long as we have the necessary knowledge and skills.

The power of the mind is also a result of the brain's ability to adapt to its environment. This is why we are able to perform tasks that are often considered to be beyond the capabilities of machines, as long as we have the necessary knowledge and skills.

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line 8 on page 8-139: The equation on line 8 is less specific than equation (*) on line 5 but has the advantage that it embodies less irrelevant information than does (*). What we wish to show is

just that, for large values of 'n', $\sum_{p=1}^n \frac{9}{10^p}$ differs very little from 1,

and we are not, at this point, interested in the fact that it is less than 1. Students may recall that the absolute value concept was introduced in Unit 1 to answer the question: 'What is the "distance" between two real numbers?'. The equation on line 8 says that the

distance between $\sum_{p=1}^n \frac{9}{10^p}$ and 1 is $(1/10)^n$.

lines 9 - 14: Students should actually test the assertion by choosing some small values of 'y' and computing the corresponding values of 'n'. For example choose 10^{-8} for 'y'. Then, "by inspection",

$$\left(\frac{1}{10}\right)^n < 10^{-8} \text{ if } n \geq 9.$$

Theorem 165 gives a much poorer estimate:

$$\left(\frac{1}{10}\right)^n < 10^{-8} \text{ if } n \geq \frac{1}{10^{-8} \left(1 - \frac{1}{10}\right)} = \frac{10^9}{9}.$$

But, the fact that this result is weaker than that found "by inspection" is unimportant. What is important is that Theorem 165 guarantees that, for each value of 'y', the inequation:

$$\left(\frac{1}{10}\right)^n < y$$

is satisfied by all sufficiently large values of 'n'. How large these values must be is, for our purposes, unimportant.

Be sure students understand why the assertion in the last line is "less explicit" than the preceding one. They can be helped to this understanding

The results of the first series of experiments are shown in Table I. The first column gives the number of trials, the second column the number of correct responses, the third column the number of errors, and the fourth column the percentage of correct responses. The results show that the percentage of correct responses increases with the number of trials.

$$P = \frac{1}{n} \sum_{i=1}^n \frac{1}{1 + \frac{1}{2^i}}$$

For a given value of n , the value of P is determined by the value of n . The value of P is always less than 1, and it approaches 1 as n increases.

$$P = \frac{1}{n} \sum_{i=1}^n \frac{1}{1 + \frac{1}{2^i}} \approx \frac{1}{n} \sum_{i=1}^n \frac{1}{1 + \frac{1}{2^i}}$$

The results of the second series of experiments are shown in Table II. The first column gives the number of trials, the second column the number of correct responses, the third column the number of errors, and the fourth column the percentage of correct responses. The results show that the percentage of correct responses increases with the number of trials.

The text on page 8-141 may with profit, be compared, almost line for line, with that on page 8-139. In particular, the reason for passing from the equation in line 6 on page 8-141 to the less explicit equation on line 8 is more apparent there than it is in the case of the equations on lines 5 and 8 of page 8-139. [But, the step is equally necessary in both cases--to say [on the earlier page] that

$$1 - \sum_{p=1}^n \frac{9}{10^p} < y,$$

for some positive number y [however small] would tell nothing about

the distance between 1 and $\sum_{p=1}^n \frac{9}{10^p}$.]

Students will probably grant the statement made in line 7 on page 8-141 without discussion. For even n , $\left(-\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^n$ and, for odd n , $\left(-\frac{1}{2}\right)^n = -\left(\frac{1}{2}\right)^n$. In either case, $\left(\frac{1}{2}\right)^n$ and $\left(-\frac{1}{2}\right)^n$ are the same distance from 0. [The corresponding step is settled in the proof, on page 8-146, of Theorem 168 by the use of Exercise 5 of Part E on page 8-145.]

Answers for Part A.

[It is a good idea to have all students do at least one of the three exercises in Part A in the manner illustrated on pages 8-138 through 8-141. The experience they have had working our preparatory exercises on exponential sequences will enable them to guess the answers very quickly. They should do at least one exercise quite carefully in order to gain an appreciation of what is involved.]

1. We are trying to find a number l such that

$$\sum_{p=1}^{\infty} \frac{3}{10^p} = l.$$

By definition, this last equation holds if and only if l is a number such that

$$\forall y > 0 \exists m \forall n \geq m \left| l - \sum_{p=1}^n \frac{3}{10^p} \right| < y.$$

Now, by Theorem 167,

$$\begin{aligned} \sum_{p=1}^n \frac{3}{10^p} &= \sum_{p=1}^n \frac{3}{10} \left(\frac{1}{10} \right)^{p-1} \\ &= \frac{3}{10} \left(\frac{1 - \left(\frac{1}{10} \right)^n}{1 - \frac{1}{10}} \right) \\ &= \frac{\frac{3}{10}}{1 - \frac{1}{10}} - \frac{\frac{3}{10} \left(\frac{1}{10} \right)^n}{1 - \frac{1}{10}} \\ &= \frac{1}{3} - \frac{1}{3} \left(\frac{1}{10} \right)^n. \end{aligned}$$

From this it follows that

$$\frac{1}{3} - \sum_{p=1}^n \frac{3}{10^p} = \frac{1}{3} \left(\frac{1}{10} \right)^n.$$

So, since $\frac{1}{3} > 0$ and, by Theorem 152a, $\left(\frac{1}{10}\right)^n > 0$,

$$\left| \frac{1}{3} - \sum_{p=1}^n \frac{3}{10^p} \right| = \frac{1}{3} \left(\frac{1}{10} \right)^n.$$

Now, look at Theorem 165. The relevant instance here is:

$$\left(0 < \frac{1}{10} < 1 \text{ and } n \geq \frac{1}{3c \left(1 - \frac{1}{10}\right)} \right) \Rightarrow \left(\frac{1}{10} \right)^n < 3c$$

['3c' because we wish the conclusion to be ' $\frac{1}{3} \left(\frac{1}{10} \right)^n < c$ '.]

So, since $0 < 1/10 < 1$,

$$\left| \frac{1}{3} - \sum_{p=1}^n \frac{3}{10^p} \right| < c \text{ if } n \geq \frac{10}{27c}.$$

Hence [using the cofinality principle--see COMMENTARY for page 8-139],

$$\forall y > 0 \exists m \forall n \geq m \left| \frac{1}{3} - \sum_{p=1}^n \frac{3}{10^p} \right| < y.$$

Consequently, $\sum_{p=1}^{\infty} \frac{3}{10^p} = \frac{1}{3}$

2. 1 [By Theorem 167c, $\sum_{p=1}^n \left(\frac{1}{2}\right)^p = 1 - \left(\frac{1}{2}\right)^n$. Since $0 < \frac{1}{2} < 1$, it follows

by Theorem 165 that, for $c > 0$, $(1/2)^n < c$ if $n \geq 2/c$. The relevant instance of Theorem 165 is:

$$\left(0 < \frac{1}{2} < 1 \text{ and } n \geq \frac{1}{c \left(1 - \frac{1}{2}\right)} \right) \Rightarrow \left(\frac{1}{2} \right)^n < c]$$

$$\frac{1}{1-x} = \frac{1}{1-x^2} + \frac{x^2}{1-x^2}$$

Let us now consider the case in which the function $f(x)$ is a polynomial.

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

Then the function $f(x)$ can be written in the form

$$f(x) = \frac{1}{1-x} + \frac{1}{1-x^2} + \dots + \frac{1}{1-x^n}$$

Let us now consider the case in which the function $f(x)$ is a rational function.

$$f(x) = \frac{P(x)}{Q(x)}$$

$$\frac{1}{1-x} = \frac{1}{1-x^2} + \frac{x^2}{1-x^2}$$

Let us now consider the case in which the function $f(x)$ is a transcendental function.

$$f(x) = \frac{1}{1-x} + \frac{1}{1-x^2} + \dots + \frac{1}{1-x^n} + \dots$$

$$\frac{1}{1-x} = \frac{1}{1-x^2} + \frac{x^2}{1-x^2}$$

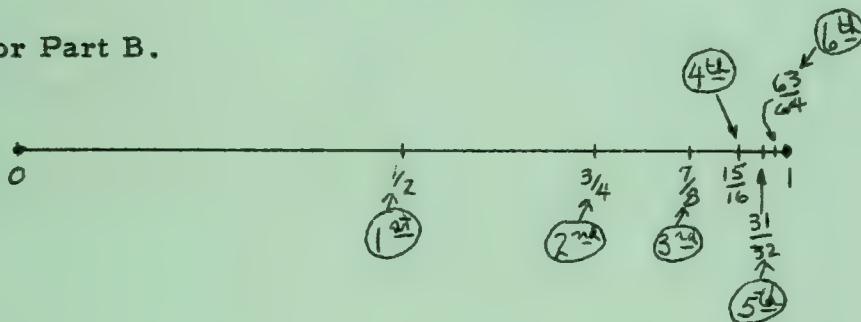
3. $\frac{1}{11}$ [The relevant instance of Theorem 165 is:

$$(0 < 0.01 < 1 \text{ and } n \geq \frac{1}{11c(1 - 0.01)}) \Rightarrow \left(\frac{1}{100}\right)^n < 11c]$$

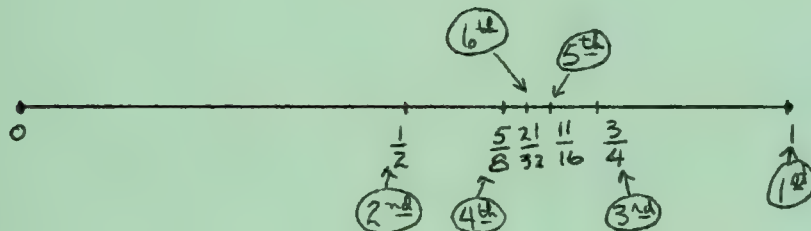
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Answers for Part B.

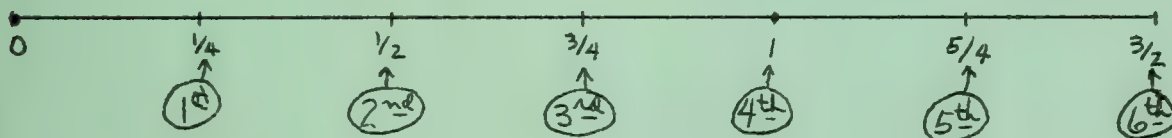
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1851 - 1852 - 1853 - 1854 - 1855 - 1856 - 1857 - 1858 - 1859 - 1860

1861 - 1862 - 1863 - 1864 - 1865 - 1866 - 1867 - 1868 - 1869 - 1870

1871

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The purpose of Part C is to point out that the terms of a continued sum sequence need not differ from the limit of the sequence.

*

Answers for Part C.

1. $\frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{6}$

2. $\frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{6}$

3. 0

4. $\frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{6}$

5. $a_1 = \frac{1}{2} = b_1;$

$$\begin{aligned} b_n + a_{n+1} &= \frac{1 + (-1)^{n+1}}{2(n+1)} + \frac{(2n+3)(-1)^{n+2} - 1}{2(n+1)(n+2)} \\ &= \frac{(n+2)(1 + (-1)^{n+1}) + (2n+3)(-1)^{n+2} - 1}{2(n+1)(n+2)} \\ &= \frac{(n+2) + (n+2)(-1)^{n+1} + (2n+3)(-1)^{n+2} - 1}{2(n+1)(n+2)} \\ &= \frac{(n+2) - (n+2)(-1)^{n+2} + (2n+3)(-1)^{n+2} - 1}{2(n+1)(n+2)} \\ &= \frac{(n+1) + (n+1)(-1)^{n+2}}{2(n+1)(n+2)} \\ &= \frac{1 + (-1)^{n+2}}{2(n+2)} = b_{n+1} \end{aligned}$$

★6. Since $n+1$ is either even or odd, it follows from Theorem 150c that $1 + (-1)^{n+1}$ is either 2 or 0, but in any case, nonnegative. So, in view of Exercise 5,

$$\left| 0 - \sum_{p=1}^n a_p \right| = \frac{1 + (-1)^{n+1}}{2(n+1)}$$

The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$\begin{aligned}
 f(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} x^{n+1} \\
 &= x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \dots
 \end{aligned}$$

$$\begin{aligned}
 f'(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} (n+1) x^n \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = e^{-x} \\
 f''(x) &= -e^{-x} \\
 f'''(x) &= e^{-x} \\
 &\vdots
 \end{aligned}$$

It is easy to see that the function $f(x)$ is a solution of the differential equation

$$f''(x) + f(x) = 0$$

If n is even,

$$\frac{1 + (-1)^{n+1}}{2(n+1)} = 0.$$

If n is odd,

$$\frac{1 + (-1)^{n+1}}{2(n+1)} = \frac{2}{2(n+1)} = \frac{1}{n+1}.$$

Since $0 < \frac{1}{n+1}$,

$$\frac{1 + (-1)^{n+1}}{2(n+1)} \leq \frac{1}{n+1}.$$

[Thus, the hint is established.]

Now, by definition, $\sum_{p=1}^{\infty} a_p = 0$ if and only if

$$\forall_{y>0} \exists_m \forall_{n \geq m} \left| 0 - \sum_{p=1}^n a_p \right| < y.$$

In view of the hint [and Theorem 92], this last result holds if

$$\forall_{y>0} \exists_m \forall_{n \geq m} \frac{1}{n+1} < y,$$

--that is, if

$$(*) \quad \forall_{y>0} \exists_m \forall_{n \geq m} n > \frac{1}{y} - 1.$$

By the cofinality principle, we know that, for each y , we can find and m such that $m > \frac{1}{y} - 1$. So, with the help of Theorem 92, $(*)$ is established.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = 1$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = 1$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = 1$$

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$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = 1$$

The above results are obtained by using the fact that the integral of the Gaussian function over the entire real line is equal to the square root of 2π . This is a well-known result in probability theory and statistics.

The sum of the series $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is 2. This is a geometric series with first term 1 and common ratio $\frac{1}{2}$. The sum of the series is given by the formula $\frac{1}{1 - \frac{1}{2}} = 2$.

2

The sum of the series $\sum_{n=0}^{\infty} \frac{1}{3^n}$ is $\frac{3}{2}$. This is a geometric series with first term 1 and common ratio $\frac{1}{3}$. The sum of the series is given by the formula $\frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$.

3

The sum of the series $\sum_{n=0}^{\infty} \frac{1}{4^n}$ is $\frac{4}{3}$. This is a geometric series with first term 1 and common ratio $\frac{1}{4}$. The sum of the series is given by the formula $\frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$.

The sum of the series $\sum_{n=0}^{\infty} \frac{1}{5^n}$ is $\frac{5}{4}$. This is a geometric series with first term 1 and common ratio $\frac{1}{5}$. The sum of the series is given by the formula $\frac{1}{1 - \frac{1}{5}} = \frac{5}{4}$.

The sum of the series $\sum_{n=0}^{\infty} \frac{1}{6^n}$ is $\frac{6}{5}$. This is a geometric series with first term 1 and common ratio $\frac{1}{6}$. The sum of the series is given by the formula $\frac{1}{1 - \frac{1}{6}} = \frac{6}{5}$.

4

The sum of the series $\sum_{n=0}^{\infty} \frac{1}{7^n}$ is $\frac{7}{6}$. This is a geometric series with first term 1 and common ratio $\frac{1}{7}$. The sum of the series is given by the formula $\frac{1}{1 - \frac{1}{7}} = \frac{7}{6}$.

The sum of the series $\sum_{n=0}^{\infty} \frac{1}{8^n}$ is $\frac{8}{7}$. This is a geometric series with first term 1 and common ratio $\frac{1}{8}$. The sum of the series is given by the formula $\frac{1}{1 - \frac{1}{8}} = \frac{8}{7}$.

5

The sum of the series $\sum_{n=0}^{\infty} \frac{1}{9^n}$ is $\frac{9}{8}$. This is a geometric series with first term 1 and common ratio $\frac{1}{9}$. The sum of the series is given by the formula $\frac{1}{1 - \frac{1}{9}} = \frac{9}{8}$.

is $\frac{1}{n} \cdot \frac{n}{2}$ if n is even. In this case, it is easy to see [and not hard to prove] that $\lim_{n \rightarrow \infty} (\bar{s})_n = \frac{1}{2}$. So, according to the definition in question, the sum of the GP with $a_1 = 1$ and $r = -1$ is $\frac{1}{2}$.

*

A proof of Theorem 168 is given on page 8-146.

*

Answers for Part D.

1. 7 $[a_1 = 7/2, r = 1/2]$

2. $3/5$ $[a_1 = 1, r = -2/3]$

3. $1/9$ $[a_1 = 1/10, r = 1/10]$

*

In doing Exercises 4-11, students should begin by using the technique shown in the Solution for the Sample. Once they see what is going on, they may skip the first few steps, and begin with, say:

$$1.\overline{246} = 1.2 + \frac{0.046}{0.99}$$

*

Answers to the rest of Part D are in the COMMENTARY for page 8-145.

The behavior of continued sum sequences of geometric progressions has been described in the COMMENTARY for page 8-140. In general, a sequence whose terms are all positive [for example, a GP whose first term and common ratio are both positive] has a continued sum sequence whose terms increase. If, in this case, the set of continued sums is bounded then the continued sum sequence has a limit, which, by definition is the sum of [all the terms of] the given sequence. [The proof of the theorem that a bounded increasing sequence has a limit depends on the completeness principle for the real numbers given in Unit 9. In fact, the theorem itself is an alternative form of the completeness principle.] If the set of continued sums is not bounded then the continued sum sequence has no limit and, consequently, by our definition of 'sum' the given sequence has no sum. [With slight modifications, these remarks apply to a sequence which has only a finite number of non positive terms.] Similar remarks, apply to a sequence all [or all but finitely many] of whose terms are negative [for example, to a GP whose first term is negative and whose common ratio is positive].

The reason for the phrase 'by our definition of 'sum'', above, is that there are many ways of defining the sum of all the terms of a sequence. The one we use is the simplest, and is adequate for at least all undergraduate college mathematics. And, as a matter of fact, for all current definitions of 'sum', a GP whose common ratio is not less than 1 has no sum.

A sequence which has infinitely many positive terms and infinitely many negative terms [for example, a GP whose common ratio is negative] has a continued sum sequence whose terms oscillate. Such a sequence may or may not have a sum in our sense [in the case of a GP with $r < 0$, it does if $r > -1$ and does not if $r \leq -1$].

A sequence whose partial sums oscillate may have a sum by some definition other than the one we are using even if it does not have a sum in our sense. For example, a GP with common ratio -1 has, by many definitions [but, not by ours], the sum $a_1/2$. According to one such definition, the sum of a sequence a is a number l if, where s is the continued sum sequence of a , $\lim_{n \rightarrow \infty} (\bar{s})_n = l$. For the GP with $a_1 = 1$ and $r = -1$ [see the last paragraph on page 8-143], $s_p = 1$ if p is odd and $= 0$

if p is even. Hence, since $(\bar{s})_n = \frac{1}{n} \sum_{p=1}^n s_p$, $(\bar{s})_n$ is $\frac{1}{n} \cdot \frac{n+1}{2}$ if n is odd and

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Theorem 86c, $b > -b$, it follows that $b + b > 0$ --that is, that $2b > 0$. So, since $2 > 0$, $b > 0$. Hence, by Theorem 98b, since $b^2 > |a|^2$, $b > |a|$. Hence, if $-b < a < b$ then $|a| < b$. Consequently, $\forall_x \forall_y \dots$

*

Note that the theorems proved in Exercises 6 and 7, together, yield 169b. [Theorem 169a is the theorem of Exercise 3, and Theorem 169c is a consequence of the theorems of Exercises 8 and 9.]

Note, also, that from Theorem 169b [and the atpi] it follows that

$$|x - a| < b \iff a - b < x < a + b$$

--that is, that $|x - a| < b$ if and only if x belongs to the interval of length $2b$ and with a as midpoint.

*

8. Note first, that if $a = c$ or $a = -c$ then $|a| = |c|$ and, so, if $c \geq 0$, $|a| = c$. From this and the theorem of Exercise 7 it follows that if $-c \leq a \leq c$ then $|a| \leq c$.

So, it will follow that $|a + b| \leq |a| + |b|$ if we can show that $-(|a| + |b|) \leq a + b \leq |a| + |b|$. This follows, by the atpi, from the fact that $\forall_x -|x| \leq x \leq |x|$. To prove this last theorem it is sufficient to note that, by definition, either $a = |a|$ or $a = -|a|$ and that, by Exercise 1, $-|a| \leq |a|$.

9. Since $a + b + -b = a$, it follows from the theorem of Exercise 8 that $|a| \leq |a + b| + |-b|$. Since $|-b| = |b|$, it follows that $|a| - |b| \leq |a + b|$. Consequently, $\forall_x \forall_y \dots$

*

The theorems of Exercises 7 - 9 are not used in the proof on page 8-146. However they form the basis of much work with absolute values and, for this reason, students should be acquainted with them.

The theorems of Exercises 3, 5, and 6 are used on page 8-146.

paragraphs of the preceding proof, one can conclude that $\forall_x \forall_y (|x| \cdot |y| = xy \text{ or } |x| \cdot |y| = -(xy))$. Now, since, by Exercise 2, $|-(ab)| = |ab|$, it follows from the theorem just proved that, in any case, $||a| \cdot |b|| = |ab|$. But, by Exercise 1, $|a| \geq 0$ and $|b| \geq 0$ and, so, $|a| \cdot |b| \geq 0$. Hence, by definition, $||a| \cdot |b|| = |a| \cdot |b|$. So, it follows that $|a| \cdot |b| = |ab|$.

Your best procedure may be to advise students to ignore the hint.]

4. For $b \neq 0$, $\frac{a}{b} \cdot b = a$ and, by Exercise 3, $|a/b| \cdot |b| = |a|$. Since, by definition, either $|b| = b$ or $|b| = -b$, it follows [using Theorem 79] that, for $b \neq 0$, $|b| \neq 0$. Hence, for $b \neq 0$, $|a/b| = |a|/|b|$.

Consequently, $\forall_x \forall_y y \neq 0 \dots$

5. (i) Since $a^1 = a$ and $|a|^1 = |a|$, it follows that $|a^1| = |a| = |a|^1$.

(ii) Suppose that $|a^P| = |a|^P$. Since $a^{P+1} = a^P \cdot a$, it follows from Exercise 3 that $|a^{P+1}| = |a^P| \cdot |a|$. So, by the inductive hypothesis, $|a^{P+1}| = |a|^P \cdot |a| = |a|^{P+1}$. Consequently,

$$\forall_n [|a^n| = |a|^n \Rightarrow |a^{n+1}| = |a|^{n+1}].$$

(iii) From (i) and (ii) it follows, by the PMI, that $\forall_n |a^n| = |a|^n$. Consequently, $\forall_x \forall_n \dots$

6. Suppose that $|a| < b$. Since $|a| \geq 0$, it follows, by Theorem 98c, that $b^2 > |a|^2 = |a^2|$. Since, by Theorem 97a, $a^2 \geq 0$, it follows, by definition, that $|a^2| = a^2$. So, $b^2 > a^2$. Since $b > |a| \geq 0$, it follows, by Theorem 98b, that $b > a > -b$. Hence, if $|a| < b$ then $-b < a < b$. Consequently, $\forall_x \forall_y \dots$

7. Suppose that $-b < a < b$. Then, $b + a > 0$ and $b - a > 0$ and, so, $b^2 - a^2 > 0$. Hence $b^2 > a^2 = |a^2| = |a|^2$. Since, by hypothesis and

$$\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d}{dt} \right)$$

$$\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d}{dt} \right)$$

$$\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d}{dt} \right) \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d}{dt} \right)$$

$$\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d}{dt} \right) \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d}{dt} \right)$$

$$\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d}{dt} \right) \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d}{dt} \right)$$

$$\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d}{dt} \right) \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d}{dt} \right)$$

$$\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d}{dt} \right) \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d}{dt} \right)$$

$$\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d}{dt} \right) \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d}{dt} \right)$$

$$\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d}{dt} \right) \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d}{dt} \right)$$

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4. $\frac{118}{55} \quad [= 2.1 + \frac{0.045}{1-0.01}]$ 5. $\frac{7}{99} \quad [= \frac{0.07}{1-0.01}]$
6. $\frac{236}{999} \quad [= \frac{0.236}{1-0.001}]$ 7. $\frac{2360}{999} \quad [= \frac{2.36}{1-0.001}]$
8. $92\frac{8}{9} \quad [= 92 + \frac{0.8}{1-0.1}]$ 9. $\frac{1}{7} \quad [= \frac{0.142857}{1-0.000001}]$
10. $\frac{10129}{1125} \quad [= 9.003 + \frac{0.0005}{1-0.1}]$ 11. $\frac{11254}{1125} \quad [= 1 + \frac{10129}{1125} \text{ (See Exercise 10.)}]$

*

Answers for Part E.

1. For $a \geq 0$, by definition, $|a| = a \geq 0$; for $a \leq 0$, by definition, $|a| = -a$ and, by Theorems 80 and 85, $-a \geq 0$. Since, by Theorem 86a, $a \geq 0$ or $0 \geq a$, it follows that $\forall_x |x| \geq 0$.
2. For $a \geq 0$, $-a \leq 0$ and, by definition, $|-a| = -(-a) = a = |a|$; for $a \leq 0$, $-a \geq 0$ and, by definition, $|-a| = -a = |a|$. Since,
3. For $ab \geq 0$, either $a \geq 0$ and $b \geq 0$ or $a \leq 0$ and $b \leq 0$. In the first case, by definition, $|a| = a$ and $|b| = b$ and, so, $|a| \cdot |b| = ab$. In the second case, by definition, $|a| = -a$ and $|b| = -b$ and, so, $|a| \cdot |b| = -a \cdot -b = ab$. So, for $ab \geq 0$, $|a| \cdot |b| = ab = |ab|$.

For $ab \leq 0$, either $a \geq 0$ and $b \leq 0$ or $a \leq 0$ and $b \geq 0$. In the first case, by definition, $|a| = a$ and $|b| = -b$ and, so, $|a| \cdot |b| = a \cdot -b = -(ab)$. In the second case, by definition $|a| = -a$ and $|b| = b$ and, so, $|a| \cdot |b| = -a \cdot b = -(ab)$. So, for $ab \leq 0$, $|a| \cdot |b| = -(ab) = |ab|$.

Consequently, $\forall_x \forall_y |x| \cdot |y| = |xy|$.

[The hint is somewhat misleading but can be made use of in the following way: Ignoring the ' $= |ab|$ ' at the end of each of the first two

line 1: The answer for 'Why?' is 'Theorem 153 [and Theorem 49 and the cpm]'.

line 3: Note the use of the theorem of Exercise 5 of Part E on page 8-145.

line 6: Note the use of the theorem of Exercise 6 of Part E.

line 4 from bottom of page: By the mtpi, it follows that if

$$\left| \frac{1}{1-r} - \sum_{p=1}^n r^{p-1} \right| < \frac{y}{|a_1|}$$

then

$$|a_1| \cdot \left| \frac{1}{1-r} - \sum_{p=1}^n r^{p-1} \right| < y.$$

But, by the theorem of Exercise 3 of Part E,

$$\begin{aligned} |a_1| \cdot \left| \frac{1}{1-r} - \sum_{p=1}^n r^{p-1} \right| &= \left| a_1 \left(\frac{1}{1-r} - \sum_{p=1}^n r^{p-1} \right) \right| \\ &= \left| \frac{a_1}{1-r} - \sum_{p=1}^n a_1 r^{p-1} \right|. \end{aligned} \quad \left. \vphantom{\sum_{p=1}^n} \right\} \begin{array}{l} \text{[Theorems} \\ \text{38 and 133].} \end{array}$$

last two cases:

$$\text{DID} = n \cdot 101$$

$$\text{EVE} = n \cdot m$$

$$\text{TALK} = 99 \cdot m$$

$$\text{DID} = n \cdot 303$$

$$\text{EVE} = n \cdot m$$

$$\text{TALK} = 33 \cdot m$$

In the first of these, since $E < D$, $n \neq 2$ and $n \neq 5$; and, since $E \neq K$, $n \neq 9$. So, $n = 3, 4, 6, 7$, or 8 . In the second case, since $D \leq 9$, $n = 2$ or 3 , but, since $E \neq K$, $n \neq 3$. So, $n = 2$.

Computation shows that there are no solutions of the first kind and just one solution, given above, of the second kind. For example, consider solutions of the first kind.

For $n = 3$, $\text{DID} = 303$ and $\text{EVE} = 3 \cdot m$. So, $m = 94, 84, 57$, or 47 . Hence, $\text{EVE} = 282, 252, 171$, or 141 , and $\text{TALK} = 9306, 8316, 5643$, or 4653 . In these cases D is A, A, K , or K , respectively, and this violates the conditions of the² problem.

For $n = 4$, $\text{DID} = 404$ and $\text{EVE} = 4 \cdot m$. So, $m = 73, 68, 63, 58$, or 53 . Hence, Etc.]

1. The first part of the paper

2. The second part of the paper

3. The third part of the paper

4. The fourth part of the paper

5. The fifth part of the paper

6. The sixth part of the paper

7. The seventh part of the paper

8. The eighth part of the paper

9. The ninth part of the paper

10. The tenth part of the paper

11. The eleventh part of the paper

12. The twelfth part of the paper

13. The thirteenth part of the paper

14. The fourteenth part of the paper

15. The fifteenth part of the paper

16. The sixteenth part of the paper

17. The seventeenth part of the paper

18. The eighteenth part of the paper

19. The nineteenth part of the paper

20. The twentieth part of the paper

21. The twenty-first part of the paper

22. The twenty-second part of the paper

23. The twenty-third part of the paper

24. The twenty-fourth part of the paper

25. The twenty-fifth part of the paper

26. The twenty-sixth part of the paper

*

Answers for Part ☆G.

$$(a) \frac{242}{303} = 0.\overline{7986}$$

$$(b) \frac{212}{606} = 0.\overline{3498}$$

[Since $\overline{\text{TALK}} = \frac{\text{TALK}}{9999}$ and $0 \leq T \leq 9$, it follows that $\frac{\text{EVE}}{\text{DID}} < 1$. Since $E \neq D$, it follows that $E < D$. Since $E \geq 1$, it follows that $D \geq 2$.

(a) Assume, now, that EVE and DID are relatively prime. It follows that DID is a factor of 9999 [= $3^2 \cdot 11 \cdot 101$]. So, since $D \geq 2$, DID is 303 or 909.

If DID = 909 then TALK = EVE · 11. But, from this it follows that K = E, so there are no solutions of this kind.

If DID = 303 then TALK = EVE · 33 and, since $E < D$, $E = 1$ or $E = 2$. If $E = 1$ then $K = 3 = D$. So, there are no solutions of this kind. If $E = 2$ then $L = 6$. Since $I = 0$, $E = 2$, $D = 3$, and $K = 6$, it follows that $V = 1, 4, 5, 7, 8$, or 9 . On multiplying each of the numbers 212, 242, 252, 272, 282, and 292 by 33 one finds that, to avoid duplications of digits, V must be 4.

So, if EVE and DID are relatively prime, the only solution is that given above.

(b) Assume, now, that $\text{HCF}(\text{EVE}, \text{DID}) = n > 1$. Then DID/n is a factor of 9999 and, since $n \geq 2$ and $\text{DID} \leq 989$, DID/n is 1, 3, 9, 11, 33, 99, 101 or 303. In these cases, respectively, EVE/n is $\frac{\text{TALK}}{9999}$, $\frac{\text{TALK}}{3333}$, $\frac{\text{TALK}}{1111}$, $\frac{\text{TALK}}{909}$, $\frac{\text{TALK}}{303}$, $\frac{\text{TALK}}{101}$, $\frac{\text{TALK}}{99}$, and $\frac{\text{TALK}}{33}$.

Suppose that $\text{EVE}/n = m$. Then, in the cases in question, $\text{TALK} = 9999 \cdot m, \dots, 99 \cdot m$, and $33 \cdot m$, respectively. Clearly, the condition that T, A, L, and K be different can be satisfied only in the

$$f(x) = \frac{1}{x} \ln x \quad \frac{2}{x} \ln x$$

Let $f(x) = \frac{1}{x} \ln x$ and $g(x) = \frac{2}{x} \ln x$. Then $f(x) = \frac{1}{x} \ln x$ and $g(x) = \frac{2}{x} \ln x$. The function $f(x)$ is defined for $x > 0$ and $g(x)$ is defined for $x > 0$. The function $f(x)$ is defined for $x > 0$ and $g(x)$ is defined for $x > 0$. The function $f(x)$ is defined for $x > 0$ and $g(x)$ is defined for $x > 0$.

$$f(x) = \frac{1}{x} \ln x \quad g(x) = \frac{2}{x} \ln x \quad f(x) = \frac{1}{x} \ln x \quad g(x) = \frac{2}{x} \ln x$$

$$\begin{aligned} f(x) &= \frac{1}{x} \ln x \\ g(x) &= \frac{2}{x} \ln x \\ f(x) &= \frac{1}{x} \ln x \\ g(x) &= \frac{2}{x} \ln x \end{aligned}$$

$$f(x) = \frac{1}{x} \ln x \quad g(x) = \frac{2}{x} \ln x \quad f(x) = \frac{1}{x} \ln x \quad g(x) = \frac{2}{x} \ln x$$

$$f(x) = \frac{1}{x} \ln x \quad g(x) = \frac{2}{x} \ln x \quad f(x) = \frac{1}{x} \ln x \quad g(x) = \frac{2}{x} \ln x$$

$$f(x) = \frac{1}{x} \ln x \quad g(x) = \frac{2}{x} \ln x \quad f(x) = \frac{1}{x} \ln x \quad g(x) = \frac{2}{x} \ln x$$

$$f(x) = \frac{1}{x} \ln x \quad g(x) = \frac{2}{x} \ln x \quad f(x) = \frac{1}{x} \ln x \quad g(x) = \frac{2}{x} \ln x$$

$$f(x) = \frac{1}{x} \ln x \quad g(x) = \frac{2}{x} \ln x \quad f(x) = \frac{1}{x} \ln x \quad g(x) = \frac{2}{x} \ln x$$

$$f(x) = \frac{1}{x} \ln x$$

$$f(x) = \frac{1}{x} \ln x \quad g(x) = \frac{2}{x} \ln x \quad f(x) = \frac{1}{x} \ln x \quad g(x) = \frac{2}{x} \ln x$$

$$f(x) = \frac{1}{x} \ln x$$

$$4. \frac{5}{6} \quad \left[\frac{3}{1-r} = 18 \right]$$

$$5. \quad 2 \text{ or } 4 \quad [a_1 + a_1 r = 9(a_1 r^2 + a_1 r^3), \frac{a_1}{1-r} = 3, \text{ and } |r| < 1 \text{ (since the progression has a sum). Since } a_1 \neq 0, \text{ it follows from the first equation that either } 1+r=0 \text{ or } 9r^2=1. \text{ From the in-equation, } r \neq -1. \text{ So, } r = 1/3 \text{ or } r = -1/3. \text{ From the second equation, } a_1 = 2 \text{ or } a_1 = 4.]$$

$$6. \quad 32/13 \quad \left[\sum_{p=1}^{\infty} a_1 r^{p-1} = 2 = \sum_{p=1}^{\infty} (a_1 r^{p-1})^2 = \sum_{p=1}^{\infty} a_1^2 (r^2)^{p-1}. \text{ So, } \frac{a_1}{1-r} = 2 = \frac{a_1^2}{1-r^2}. \text{ Since } |r| < 1, \text{ it follows that } r^2 \neq 1, \text{ and } a_1 = 1+r. \text{ Hence } \frac{1+r}{1-r} = 2, \text{ and } r = 1/3. \text{ So, } a_1 = 4/3. \right]$$

$$\text{Consequently, } \sum_{p=1}^{\infty} (a_1 r^{p-1})^3 = \frac{a_1^3}{1-r^3} = \frac{32}{13}.]$$

$$7. \quad 3/2 \text{ and } 1/3 \quad [a_1 + a_1 r = 2 \text{ and } \forall_n a_n = 2 \sum_{p=1}^{\infty} (a_1 r^n) r^{p-1}. \text{ It follows}$$

$$\text{from the second condition that } a_1 = 2 \sum_{p=1}^{\infty} a_1 r^p = 2r \sum_{p=1}^{\infty} a_1 r^{p-1}$$

$$= 2r \frac{a_1}{1-r}. \text{ Since } a_1 \neq 0, r = 1/3 \text{ and, since } a_1 = \frac{2}{1+r}, a_1 = 3/2. \text{ To complete the solution it is necessary to show that the given conditions are satisfied. In particular, for each } n,$$

$$2 \sum_{p=1}^{\infty} \frac{3}{2} \left(\frac{1}{3} \right)^{n+p-1} = 2 \cdot \frac{3}{2} \left(\frac{1}{3} \right)^n \cdot \frac{1}{1-\frac{1}{3}} = \frac{3}{2} \left(\frac{1}{3} \right)^{n-1} = a_n.]$$

following way. The number s of feet travelled in t seconds by a body falling from rest is, approximately $16t^2$. So, $t \doteq \frac{1}{4}\sqrt{s}$. By symmetry, the upward part of each bounce takes the same time as the downward part. So, neglecting the periods during which the bottom point of the ball is in contact with the ground (when the ball is subject to elastic forces in addition to the force of gravitation), the time (in seconds) that the ball is in motion is, approximately,

$$\begin{aligned} \frac{1}{4}\sqrt{50} + 2 \sum_{p=1}^{\infty} \frac{1}{4}\sqrt{50\left(\frac{4}{5}\right)^p} &= \frac{\sqrt{50}}{4} \left(1 + \frac{4}{\sqrt{5}} \sum_{p=1}^{\infty} \left(\frac{2}{\sqrt{5}} \right)^{p-1} \right) \\ &= \frac{\sqrt{50}}{4} \left(1 + \frac{4}{\sqrt{5}} \frac{1}{1 - \frac{2}{\sqrt{5}}} \right) = \frac{5\sqrt{2}}{4} \left(1 + \frac{4}{\sqrt{5} - 2} \right) = \frac{5\sqrt{2}}{4} \frac{\sqrt{5} + 2}{\sqrt{5} - 2} \\ &= \frac{5\sqrt{2}(\sqrt{5} + 2)^2}{4} = \frac{5\sqrt{2}(9 + 4\sqrt{5})}{4} \doteq 31.7. \end{aligned}$$

So, it seems a safe guess that the ball will bounce for less than one minute.]

3. $256\sqrt{3}$ [The area-measure of an equilateral triangle inscribed in a circle of radius r is $\frac{3\sqrt{3}}{4}r^2$. Also, if the circumradius of an equilateral triangle is r then the inradius is $r/2$. So, the area-measures of the successive triangles are

$$\frac{3\sqrt{3}}{4} \cdot 16^2, \quad \frac{3\sqrt{3}}{4} \cdot \left(\frac{16}{2}\right)^2, \quad \frac{3\sqrt{3}}{4} \cdot \left(\frac{16}{4}\right)^2, \quad \dots$$

The required sum is computed as follows:

$$\frac{3\sqrt{3}}{4} \cdot 16^2 \sum_{p=1}^{\infty} \left(\frac{1}{4} \right)^{p-1} = \frac{3\sqrt{3}}{4} \cdot 16^2 \cdot \frac{1}{1 - \frac{1}{4}} = \sqrt{3} \cdot 16^2 = 256\sqrt{3}.]$$

Answers for Part F.

1. (a) 18 (b) $\frac{50}{9}$ (c) $\frac{35}{8}$ (d) $\frac{7\sqrt{2}(\sqrt{7}+1)}{6}$

2. 286.16 (feet); 450 (feet)

[To make sense of this problem one should assume that distances are measured from the bottom point of the ball. (If, for example, the top of the ball is initially 50 feet from the ground, the top will travel more than 50 feet before starting up again.) One should also assume that the surface on which the ball strikes is completely rigid, so that it is not depressed when the ball strikes. (Otherwise, even the bottom point of the ball will travel more than 50 feet before starting up.) With these assumptions, the distance (in feet) that the ball has moved by the time it hits the ground the fifth time is

$$\begin{aligned} & 50 + 2\left(50 \cdot \frac{4}{5}\right) + 2\left(50 \cdot \left(\frac{4}{5}\right)^2\right) + 2\left(50 \cdot \left(\frac{4}{5}\right)^3\right) + 2\left(50 \cdot \left(\frac{4}{5}\right)^4\right) \\ &= 50 + 2 \sum_{p=1}^4 50 \left(\frac{4}{5}\right)^p = 50 \left(1 + \frac{8}{5} \sum_{p=1}^4 \left(\frac{4}{5}\right)^{p-1}\right) = 50 \left(1 + \frac{8}{5} \frac{1 - \left(\frac{4}{5}\right)^4}{1 - \frac{4}{5}}\right) \\ &= 50(1 + 8 \times 0.5904) = 286.16. \end{aligned}$$

The second part of the problem requires additional assumptions. Actually, much of the energy of the ball will be dissipated by internal and external frictional forces, and the ball will come to rest after some finite number of bounces. However, assuming this not to be the case, one can compute that the total distance (in feet) the ball will move is

$$\begin{aligned} & 50 + 2 \sum_{p=1}^{\infty} 50 \left(\frac{4}{5}\right)^p = 50 \left(1 + \frac{8}{5} \sum_{p=1}^{\infty} \left(\frac{4}{5}\right)^{p-1}\right) = 50 \left(1 + \frac{8}{5} \frac{1}{1 - \frac{4}{5}}\right) \\ &= 50(1 + 8) = 450. \end{aligned}$$

(The additional assumption--neglect of frictional forces--is often made in order to obtain a simple approximation to the solution of a physical problem.)

Students may ask how one knows that the ball will ever stop. Under our idealizing assumptions this can be partially answered in the

In view of the text on pages 8-106 and 8-107 and the accompanying COMMENTARY, pages 8-148 through 8-152 should require little additional explanation. The earlier development of base- m representations is reviewed at length from the lower half of page 8-149 through the upper third of page 8-151. The new material, on base- m approximations, parallels this very closely as will be apparent on comparing the displays on page 8-150 with the similar ones on pages 8-151 and 8-152.

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The positive numbers which have terminating decimal representations are those which are quotients of positive integers by nonnegative integral powers of ten--somewhat more sophisticatedly, they are those which are quotients of positive integers by positive integers which have no prime factors other than 2 and 5.

Similarly, for each $m > 1$, the positive numbers which have terminating base- m representations are those which are quotients of positive integers by nonnegative integral powers of m --that is, by positive integers which have no prime factors other than those of m .

Since, for example, $1.75 = 1.75\bar{0}$, it is proper to omit the '[or terminating]' on lines 9 and 10 from the bottom of page 8-148.

*

The "slight modification" referred to in lines 10 and 11 on page 8-150 consists of starting at 0, rather than at 1.

*

The last line on page 8-151 is obtained from the preceding by noticing that, by definition,

$$cm^q = \ll cm^q \gg + \{\{ cm^q \}\}.$$

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line 1: If $0 < c < m^p$ then $0 < c/m^p < 1$ and $\llbracket c/m^p \rrbracket = 0$. Notice that this is the first [in fact, the only] place in the proof where the assumption that $c > 0$ is used.

lines 5 and 6: Since, by Theorem 151b, $m^k > k$ if $k \geq 0$, it follows that if $k \in I^+$ and $k \geq c$ then $m^k > c$. For example, for $c > 0$, if $k = \llbracket c \rrbracket + 1$ then $k \in I^+$ and $c < m^k$.

lines 8 and 9: If $0 < c < 1$ then, since $m > 1$, $c < m^1$. Since 1 is the least positive integer, it, even more so, is the least positive integer k such that $c < m^k$. In this case, $\llbracket c/m^{p-1} \rrbracket = \llbracket c/m^0 \rrbracket = \llbracket c \rrbracket = 0$.

next-to-last line: 'the preceding computation' refers to the next-to-last line on page 8-151, subject to the choice of p such that $\llbracket c/m^p \rrbracket = 0$ [see first line on page 8-152].

1. The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \frac{1}{x} \int_0^x f(t) dt.$$

$$f'(x) = -\frac{1}{x^2} \int_0^x f(t) dt + \frac{1}{x} f(x).$$

It is shown that the function $f(x)$ is continuous and differentiable for all $x \neq 0$.

2. The second part of the paper is devoted to the study of the properties of the function $g(x)$ defined by the equation

$$g(x) = \frac{1}{x} \int_0^x g(t) dt + \frac{1}{x} \int_x^\infty g(t) dt.$$

It is shown that the function $g(x)$ is continuous and differentiable for all $x \neq 0$.

3. The third part of the paper is devoted to the study of the properties of the function $h(x)$ defined by the equation

$$h(x) = \frac{1}{x} \int_0^x h(t) dt + \frac{1}{x} \int_x^\infty h(t) dt + \frac{1}{x} \int_0^\infty h(t) dt.$$

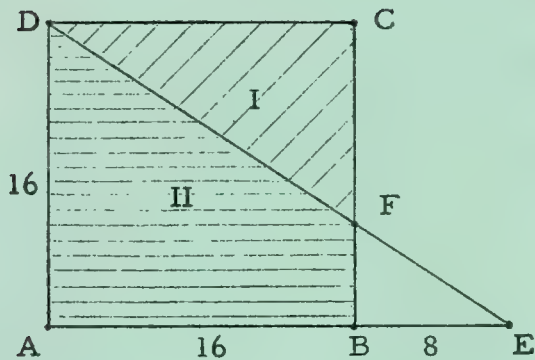
It is shown that the function $h(x)$ is continuous and differentiable for all $x \neq 0$.

4. The fourth part of the paper is devoted to the study of the properties of the function $k(x)$ defined by the equation

$$k(x) = \frac{1}{x} \int_0^x k(t) dt + \frac{1}{x} \int_x^\infty k(t) dt + \frac{1}{x} \int_0^\infty k(t) dt + \frac{1}{x} \int_0^\infty k(t) dt.$$

It is shown that the function $k(x)$ is continuous and differentiable for all $x \neq 0$.

8.



$$\frac{FB}{DA} = \frac{EB}{EA}, \quad \frac{FB}{16} = \frac{8}{24}, \quad FB = 16/3$$

$$K(I) = \frac{1}{2} \cdot 16(16 - \frac{16}{3}) = \frac{256}{3}$$

$$K(II) = 16^2 - \frac{256}{3} = \frac{512}{3}$$

9. $8/13 [4 + 3x - (9 - 2x) = 7 - 5x - (4 + 3x)]$

10. 3 [By trial-and-error it is clear that $2^4 < 4!$. But, what remains to be shown is that, for each $p > 3$, $2^p < p!$. This is easy to do by induction (actually, Theorem 114--j = 4). We know that $2^4 < 4!$. Now, suppose that [for $q > 3$], $2^q < q!$. Since $2^{q+1} = 2 \cdot 2^q$, it follows that $2^{q+1} < 2 \cdot q!$. Also, $2 < q + 1$. So, $2^{q+1} < (q+1)q! = (q+1)!.]$

11. $2x^2 + 3x = 0$

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Answers for Miscellaneous Exercises.

$$\left[\begin{array}{l} \text{Easy:} \quad 1-3, 5-16, 18-22, 24, 25, 27-33, 35, 37, 38, 41-43; \\ \text{Medium:} \quad 4, 17, 34, 39, 44; \\ \text{Hard:} \quad 23, 26, 36, 40, 45 \end{array} \right]$$

1. (a) $|x + 5|$ (b) $|11 - m^2|$ (c) $9|x^5|$

2. $2 \left[\frac{2A - B}{3A - 2B} = \frac{2 \cdot \frac{A}{B} - 1}{3 \cdot \frac{A}{B} - 2} = \frac{2 \cdot \frac{3}{4} - 1}{3 \cdot \frac{3}{4} - 2} = \frac{6 - 4}{9 - 8} = 2 \right]$

3. $V_1 = \frac{VS_1 \left(\frac{1}{d_1} + \frac{1}{d_2} \right) - 4\pi Q}{S_1 \left(\frac{1}{d_1} + \frac{1}{d_2} \right)}$

4. 54.5 $[2(x + y) = 302, xy = 5460; x^2 + y^2 = 151^2 - 2xy = 11881;$
 $\sqrt{x^2 + y^2} = 109]$

5. (a) $81x^6 + 36x^5 + 76x^4 + 124x^3 + 40x^2 + 48x + 36$
 (b) $x^8 + 5x^7 + 4x^6 + 3x^5 + 17x^4 + 22x^3 + 9x^2 + 5x + 4$

6. $\frac{(a+b)^2}{4} - \frac{(a-b)^2}{4} = \frac{[(a+b) - (a-b)][(a+b) + (a-b)]}{4}$
 $= \frac{2b \cdot 2a}{4} = ba$

7. (a) true ["Each prime (greater than 2) is an odd number."]
 (b) false ["A positive integer (greater than 2) is a prime if and only if it is an odd number." Try 9.]

12. $\frac{8}{3}$

13. 40 minutes $[\frac{20}{24 \times 1.25} \text{ hours}]$

14. Assuming that, for each n ,

$$\prod_{p=1}^n (2p-1) \cdot \prod_{p=1}^n (2p) = \prod_{p=1}^{2n} p$$

[that is, that the product of the first $2n$ positive integers is the product of the first n odd positive integers times the product of the first n even positive integers], it is easy to proceed. For, by Theorem 145,

$$\prod_{p=1}^n (2p) = \prod_{p=1}^n 2 \cdot \prod_{p=1}^n p = 2^n \cdot n!, \text{ by definition.}$$

Here is an outline of an inductive proof which avoids the assumption and Theorem 145:

$$(i) \quad \prod_{p=1}^1 (2p-1) = 2 \cdot 1 - 1 = 1; \quad \frac{(2 \cdot 1)!}{2^1 \cdot 1!} = 1$$

$$\begin{aligned} (ii) \quad \prod_{p=1}^{q+1} (2p-1) &= \prod_{p=1}^q (2p-1)(2q+1) = \frac{(2q)!}{2^q \cdot q!} (2q+1) \\ &= \frac{(2q+2)!}{(2q+2)2^q \cdot q!} = \frac{(2q+2)!}{2^{q+1} \cdot (q+1)!} \end{aligned}$$

15. Suppose that $m(\angle AOB) = a$ and $m(\angle COD) = b$. The common measure of the remaining two angles of [isosceles triangle] $\triangle AOB$ is $90 - \frac{a}{2}$, and that of the remaining two angles of $\triangle COD$ is $90 - \frac{b}{2}$. So, if, in some similarity between $\triangle AOB$ and $\triangle COD$, O does not correspond

The following is a list of the names of the members of the American Medical Association who have been elected to the office of President of the Association for the year 1911.

The following is a list of the names of the members of the American Medical Association who have been elected to the office of Secretary of the Association for the year 1911.

The following is a list of the names of the members of the American Medical Association who have been elected to the office of Treasurer of the Association for the year 1911.

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The following is a list of the names of the members of the American Medical Association who have been elected to the office of President of the Association for the year 1912.

with itself then $a = 90 - \frac{b}{2}$ and $b = 90 - \frac{a}{2}$ ["corresponding angles of similar triangles are congruent"]. From this it follows that $a = b = 60$. So, in this case, $\angle AOB \cong \angle COD$. On the other hand, if, in the postulated similarity, O corresponds with itself, then $\angle AOB \cong \angle COD$.

16. \sqrt{ab}

17. 2, 4, 6, 9 or $\frac{35}{4}, \frac{25}{4}, \frac{15}{4}, \frac{9}{4}$

$$\begin{aligned} [& \quad (1) \ a_1 + a_3 = 2a_2 \\ & \quad (2) \ a_3^2 = a_2 a_4 \\ & \quad (3) \ a_1 + a_4 = 11 \\ & \quad (4) \ a_2 + a_3 = 10 \end{aligned}$$

From (1), (3), and (4):

$$(5) \ 3a_2 + a_4 = 21$$

From (2) and (5):

$$(6) \ a_3^2 = a_2(21 - 3a_2)$$

From (4) and (6):

$$\begin{aligned} (7) \ (10 - a_2)^2 &= a_2(21 - 3a_2) \\ a_2 = 4 \quad \text{or} \quad a_2 &= \frac{25}{4} \end{aligned} \quad]$$

18. -4912 [By the quadratic formula,

$$r_1 = \frac{-9824 + \sqrt{9824^2 - 4 \cdot 2 \cdot -7}}{4}$$

and

$$r_2 = \frac{-9824 - \sqrt{9824^2 - 4 \cdot 2 \cdot -7}}{4}$$

$$\text{So,} \quad r_1 + r_2 = \frac{-9824}{4} + \frac{-9824}{4} = -\frac{9824}{2}. \quad]$$

1. The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \frac{1}{x} \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function.

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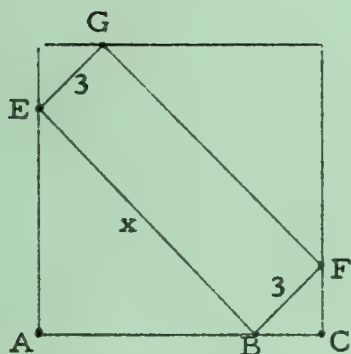
8. The eighth part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \frac{1}{x} \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function.

9. The ninth part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \frac{1}{x} \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function.

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11. The eleventh part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \frac{1}{x} \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function.

19. Yes



strip of carpeting will fit [in the position shown].

Let EBF represent a rug placed symmetrically with respect to a diagonal. Then, $BC = \frac{3}{\sqrt{2}}$ and

$AB = \frac{x}{\sqrt{2}}$. So, $\frac{x}{\sqrt{2}} + \frac{3}{\sqrt{2}} = 10$ and

$x = 10\sqrt{2} - 3$. Therefore, $x > 14.14$

$- 3 = 11.14$. Since 11 feet 1 inch =

$11\frac{1}{12}$ feet, and $11\frac{1}{12} < 11.14$, the

20. $\frac{1}{3}$, 0, $f(x)$, $\frac{x^2}{3x^2 + 7x + 7}$, $\frac{y^2 - 4y + 4}{3y^2 - 5y + 5}$

21. [Pythagorean theorem or s.a.s.]

22. 90/31

1. The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \frac{1}{2} \left(f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right) \right)$$

where $f(0) = 0$ and $f(1) = 1$. It is shown that the function $f(x)$ is continuous and increasing on the interval $[0, 1]$.

2. In the second part, it is proved that the function $f(x)$ is a solution of the functional equation

$$f(x) = \frac{1}{2} \left(f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right) \right)$$

for all $x \in [0, 1]$. The proof is based on the fact that the function $f(x)$ is a solution of the equation

[illegible]

1 -

1980-1981 yılları arasında yapılan araştırmalar, Türkiye'de yaşayan Türklerin %70'ini ilköğretim okullarında eğitildiğini göstermiştir. Bu oran, o dönemde dünya ortalaması olan %60'tan daha yüksektir. Ancak, bu oranın artmasıyla birlikte, eğitim kalitesi ve öğretmenlerin yeterliliği de tartışılmaya başlanmıştır. Özellikle kırsal alanlarda, öğretmenlerin maaşları düşük olduğu için işlerini bırakarak göç etmişlerdir. Bu durum, eğitimde eşitsizlik yaratmış ve kırsal alanların geride kalmasına neden olmuştur.

[illegible]

$$M(\mathbb{C}^n) \cong M(\mathbb{C}) \otimes M(\mathbb{C})$$

Consequently, for each n ,

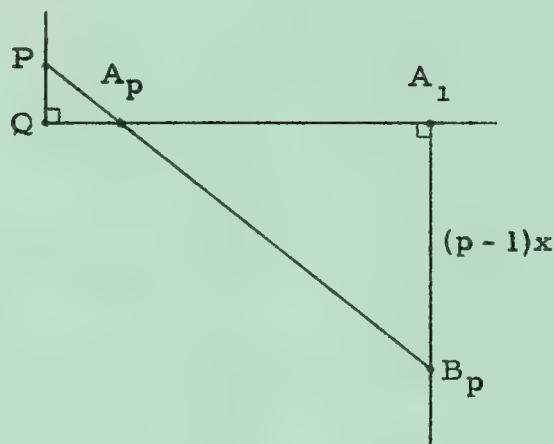
$$a_n = \sum_{p=1}^n a_p - \sum_{p=1}^{n-1} a_p = \left[\frac{n(n+1)}{2} \right]^2 - \left[\frac{n(n-1)}{2} \right]^2 = n^3.$$

$$27. \quad f(f(x)) = \frac{\frac{x}{\frac{x}{x-1} - 1}}{\frac{x}{x-1} - 1} = \frac{x}{x - (x-1)} = x$$

[Alternatively: Since $f(x) = 1 + \frac{1}{x-1}$, it follows that

$$f(f(x)) = 1 + \frac{1}{f(x) - 1} = 1 + \frac{1}{1/(x-1)} = 1 + (x-1) = x.]$$

23. (a)



Since $A_1A_pB_p \leftrightarrow QA_1P$

is a similarity,

$$\frac{A_1A_p}{QA_p} = \frac{A_1B_p}{QP} = p - 1.$$

So, $\frac{A_1A_p}{QA_p} + 1 = p$, and,

since $A_1A_p + QA_p = QA_1 = 1$,
 $1/QA_p = p$.

Hence, $QA_p = 1/p$ and, for each n , $A_1A_n = 1 - \frac{1}{n}$. Hence,

$$\forall_n \sum_{p=1}^n A_p A_{p+1} = 1 - \frac{1}{n+1}.$$

(b) [See answer for Exercise 4(b) of Part E on page 8-137.]

24. 1000 $[k + 3 = 10]$

25. (a) $\frac{2y}{x+y}$

(b) $\frac{x^3 + x^2 + x + 1}{x^2 + x + 1}$

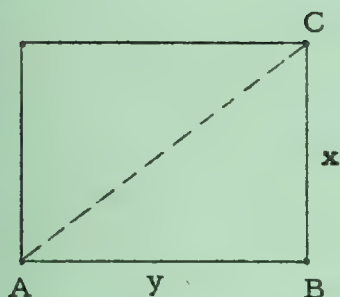
(c) $\frac{y}{(x+y)^2}$

26. The p th term of the sequence a is the sum of p consecutive odd numbers, and, in fact, the sum of the first n terms of a is the sum

of the first $\sum_{p=1}^n p$ positive odd numbers. Since $\sum_{p=1}^n p = \frac{n(n+1)}{2}$,

$$\sum_{p=1}^n a_p = \sum_{p=1}^{\frac{n(n+1)}{2}} (2p-1) = \left[\frac{n(n+1)}{2} \right]^2.$$

28.



$$x + y = 14$$

$$xy = 48$$

$$y > x$$

$$\text{So, } x = 6 \text{ and } y = 8$$

$$\text{Hence, } \tan \angle CAB = \frac{3}{4} \text{ and } \tan \angle ACB = \frac{4}{3}.$$

29. Let m be the slope of the given linear function, and let x_1, x_2, \dots, x_{10} be the first components of the given ordered pairs. Since the latter are consecutive terms of an AP, there is a number d such that, for $p \leq 9$, $x_{p+1} - x_p = d$. [Since there are 10 ordered pairs it follows that $x_2 \neq x_1$ and, so, that $d \neq 0$.] Now, if y_1, y_2, \dots, y_{10} are the corresponding second components it follows that, for $p \leq 9$,

$$\frac{y_{p+1} - y_p}{x_{p+1} - x_p} = m,$$

and, so, that $y_{p+1} - y_p = md$. Hence, the second components are consecutive terms of an AP whose common difference is md .

30. From geometry we know that if p_n is the perimeter of the n th square then $p_n \sqrt{2}$ is the perimeter of the $(n+1)$ th square. So,

$$p_{100} = 1(\sqrt{2})^{99} = 2^{49}\sqrt{2}.$$

31. (a) $\frac{y^2 + 3y - 4}{y - 2}$

(b) $\frac{7a + 2}{3a(a - 2)}$

$$\begin{aligned} 32. & (2k - 1)^2 + (2k + 1)^2 + (2k + 3)^2 \\ &= 12k^2 + 12k + 11 \\ &= 3(4k^2 + 4k + 1) + 8 \\ &= 3(2k + 1)^2 + 8 \end{aligned}$$

• $\frac{1}{2} = 0.5$ $\frac{1}{4} = 0.25$ $\frac{1}{5} = 0.2$ $\frac{1}{10} = 0.1$

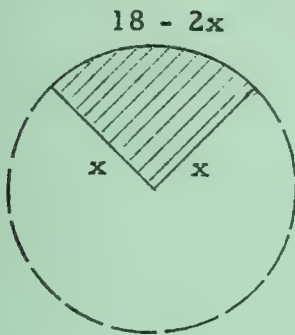
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1. 1. 1.

33. For $x \neq 0$, $x^{-1} = -1$ if and only if $x = -1$. So, for $-1 \neq x \neq 0$,

$$f(x^{-1}) = \frac{x^{-1} - 1}{x^{-1} + 1} = \frac{1 - x}{1 + x} = -\frac{x - 1}{x + 1} = -f(x).$$

34. 4.5 inches



$$\begin{aligned} K &= \left(\frac{18 - 2x}{2\pi x} \right) \pi x^2 = (9 - x)x \\ &= \frac{81}{4} - \left(x - \frac{9}{2}\right)^2 \end{aligned}$$

35. 2135 [The p th installment is $300 + 3\% (1500 - 300[p - 1])$ dollars. Since there are 5 installments [$5 \cdot 300 = 1500$], the total number of dollars paid for the car is

$$\begin{aligned} &500 + \sum_{p=1}^5 [300 + 3\% (1500 - 300[p - 1])] \\ &= 500 + 1500 + 225 - 9 \sum_{p=1}^5 (p - 1) \\ &= 2135. \end{aligned}$$

[Note the convention that '6% interest' is an abbreviation for '6% annual interest'.]

36. the 10^{2499} place [that is, the first nonzero digit counting from the right occurs in the 2500th place] [The problem is one of finding the largest n such that $10^n \mid 10000!$. Since $10^n = 2^n \cdot 5^n$

$$f(x) = \frac{1}{x^2} = x^{-2} \Rightarrow f'(x) = -2x^{-3} = -\frac{2}{x^3}$$

$$f'(x) = -\frac{2}{x^3}$$

$$f(x) = \frac{1}{x^2} = x^{-2} \Rightarrow f'(x) = -2x^{-3} = -\frac{2}{x^3}$$

$$f'(x) = -\frac{2}{x^3}$$

$$f(x) = \frac{1}{x^2} = x^{-2} \Rightarrow f'(x) = -2x^{-3} = -\frac{2}{x^3}$$

$$f\left(\frac{1}{2}\right) = \frac{1}{\left(\frac{1}{2}\right)^2} = 4$$

$$f'(x) = -\frac{2}{x^3}$$

$$f'(x) = -\frac{2}{x^3}$$

$$f(x) = \frac{1}{x^2} = x^{-2} \Rightarrow f'(x) = -2x^{-3} = -\frac{2}{x^3}$$

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$$f'(x) = -\frac{2}{x^3}$$

$$f(x) = \frac{1}{x^2} = x^{-2} \Rightarrow f'(x) = -2x^{-3} = -\frac{2}{x^3}$$

and since there are more multiples of 2 than of 5 among the first 10000 positive integers, the problem reduces to one of finding the largest n such that $5^n \mid 10000!$. Now, since 5 is a factor of each multiple of 5 and since there are 2000 multiples of 5 among the first 10000 positive integers, it follows that

$$5^{2000} \mid 10000!$$

But, some of these multiples of 5 are also multiples of 5^2 , 5^3 , 5^4 , and 5^5 . In fact there are 400 multiples of 5^2 , 80 multiples of 5^3 , 16 multiples of 5^4 , and 3 multiples of 5^5 . So,

$$5^{2499} \mid 10000!.$$

Since $5^6 = 15625 > 10000$, 2499 is the largest n such that $5^n \mid 10000!$. So, the decimal numeral for 10000! ends in 2499 zeros.

In general, the exponent of the largest power of a prime, p , which divides

$$n! \text{ is } \sum_{k=1}^m \left\lfloor \frac{n}{p^k} \right\rfloor, \text{ where } m \text{ is the least integer such that } p^{m+1} > n.]$$

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$$\frac{1}{2} \frac{b^2}{a} = \frac{b^2}{2a}$$

$$S_1 = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2}$$

[illegible]

43. $88\frac{8}{9}$ [If d is the diameter of the inner circle then $3d$ is the diameter of the outer circle.

$$\frac{\pi \left(\frac{3d}{2}\right)^2 - \pi \left(\frac{d}{2}\right)^2}{\pi \left(\frac{3d}{2}\right)^2} = \frac{8}{9}]$$

44. $c^2 + c + 1 = \left(c + \frac{1}{2}\right)^2 + \frac{3}{4} \geq 0 + \frac{3}{4} > 0$

45. $C(n, 4)$ --that is [see page 7-71], the number of 4-membered subsets of an n -membered set

[Each 4 of the n points are end points of two intersecting chords. The problem of computing values of ' $C(n, 4)$ ' is taken up on page 8-168.]

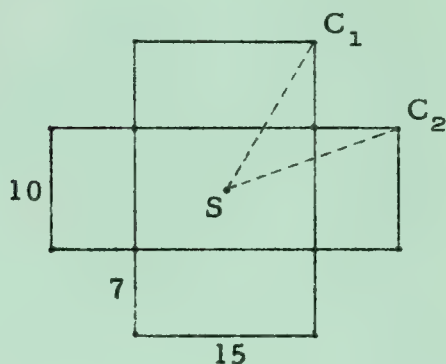
37. If there were numbers x and y such that $x^2 - 16y^2 = 10$ and $x = 4y$ then, for such numbers, $(4y)^2 - 16y^2 = 10$ --that is, $0 = 10$. But, $0 \neq 10$.

38. $5 [6n + 3 + 2(3n + 6) = 3n \cdot n \iff (n = 5 \text{ or } n = -1); n > 0]$

39. (a) a (b) $\frac{x^2 y^3}{x + y}$

40. approximately 14 feet $[\frac{1}{2}\sqrt{801}]$

[Consider the walls and floor as if they were subsets of the same plane.]



$$SC_1 = \sqrt{\left(\frac{15}{2}\right)^2 + (7 + 5)^2} = \frac{1}{2}\sqrt{801};$$

$$SC_2 = \sqrt{(5)^2 + \left(7 + \frac{15}{2}\right)^2} = \frac{1}{2}\sqrt{941}$$

The shortest path to any of the upper corners is SC_1 feet long.]

41. [Theorem 156]

42. $AP = \frac{y - x}{2}$ [We are looking for a point $P \in \overline{AB}$ such that the area-measure of the trapezoidal region bounded by APCD is the area-measure of the triangular region bounded by PBC. So, P is a point such that

$$\frac{1}{2}h(x + AP) = \frac{1}{2}h(y - AP).$$

Hence, $AP = \frac{y - x}{2}$.]

Intermediate steps in Examples 1(b) and 1(d) might help:

difference of squares

$$(b) \quad 4u^2 - \frac{v^2}{9} = (2u)^2 - \left(\frac{v}{3}\right)^2 =$$

$$(d) \quad a^4 - 16d^4 = (a^2)^2 - (4d^2)^2 =$$

difference of squares

Naturally, students should not be encouraged to write these intermediate steps in their own work. But, pedagogically, it is handy to have such a technique around when you find a student who has difficulty. A related technique is the use of frames:

$$\text{[frame]}^2 - \text{[frame]}^2$$

This is very useful in a problem such as Exercise 25 on page 8-159.

The explanation asked for at the end of this text is that, allowing radical signs and introducing restrictions, any difference [or sum] is a difference of squares:

$$\forall x \geq 0 \quad \forall y \geq 0 \quad x - y = (\sqrt{x})^2 - (\sqrt{y})^2 \quad [\forall x \geq 0 \quad \forall y \leq 0 \quad x + y = (\sqrt{x})^2 - (\sqrt{-y})^2]$$

Of course, as pointed out on page 4-75 ff. of Unit 4, when factoring, one does not usually introduce radical signs.]

*

Answers for Part A.

- | | | |
|---|---|---|
| 1. $(t - 1)(t + 1)$ | 2. $(2a - 3)(2a + 3)$ | 3. $(1 - 4x)(1 + 4x)$ |
| 4. $(6x - 7y)(6x + 7y)$ | 5. $\left(x - \frac{y}{5}\right)\left(x + \frac{y}{5}\right)$ | 6. $\left(4a - \frac{3b}{5}\right)\left(4a + \frac{3b}{5}\right)$ |
| 7. $\left(\frac{2}{x} - 1\right)\left(\frac{2}{x} + 1\right)$ | 8. $\left(\frac{8}{b} - \frac{1}{4c}\right)\left(\frac{8}{b} + \frac{1}{4c}\right)$ | 9. $(x^2 - yz)(x^2 + yz)$ |
| 10. $(5a^2 - 1)(5a^2 + 1)$ | 11. $(xy - 1)(xy + 1)$ | 12. $(10a - 2b^2c)(10a + 2b^2c)$ |

Answers for the rest of Part A are in the COMMENTARY for page 8-159.

$$\frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2}$$

$$\frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2}$$

1917-18-19

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$$\frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2}$$

$$\frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2}$$

$$\frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2}$$

$$\frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2}$$

$$\frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2}$$

$$13. \left(\frac{a^2 b^3}{2} - \frac{2}{c} \right) \left(\frac{a^2 b^3}{2} + \frac{2}{c} \right)$$

$$14. \left(t - \frac{s}{2} \right) \left(t + \frac{s}{2} \right) \left(t^2 + \frac{s^2}{4} \right)$$

$$15. \left(\frac{1}{c^2} - \frac{5}{d} \right) \left(\frac{1}{c^2} + \frac{5}{d} \right)$$

$$16. \left(\frac{x^2}{y} - \frac{1}{z} \right) \left(\frac{x^2}{y} + \frac{1}{z} \right) \left(\frac{x^4}{y^2} + \frac{1}{z^2} \right)$$

$$17. (x - y + 7)(x + y + 3)$$

$$18. (2x - 5y + 4)(2x + 5y - 10)$$

$$19. (2a + 3b)(4a + b)$$

$$20. 5t(5t - 2)$$

$$21. 3(3s + 1)(s - 1)$$

$$22. (7 - 5u)(7 + u)$$

$$23. (x - y - 5)(x + y - 5)$$

$$24. (a - b - 1)(a - b + 1)$$

$$25. (2x - y - 3a - b)(2x - y + 3a + b)$$

$$26. (y - 2z - 2u + v)(y - 2z + 2u - v)$$

*

The explanation asked for three lines from the foot of page 8-159 is that, since, for $a \neq 0$,

$$1 - \left(\frac{b}{a} \right)^2 = \left(1 - \frac{b}{a} \right) \left(1 + \frac{b}{a} \right)$$

and since $a^2 = aa$, it follows that, for $a \neq 0$,

$$\left(1 - \frac{b^2}{a^2} \right) a^2 = \left(1 - \frac{b}{a} \right) a \left[\left(1 + \frac{b}{a} \right) a \right].$$

*

In connection with Theorem 170 on page 8-160 it is instructive to ask students to complete:

$$x^k - y^k = (x - y)(\quad ? \quad)$$

by using the division-with-remainder algorithm.

... ..

$$f = \frac{1}{\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk$$

(1) $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, where \mathcal{C}_1 and \mathcal{C}_2 are disjoint and

$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x) e^{-x^2} dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x) e^{-x^2} dx$

$\frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2}$

$$G(\Sigma) = \sum_{\sigma \in \Sigma} G(\sigma) = \sum_{\sigma \in \Sigma} \frac{1}{|\sigma|!} \left(\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \right) \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \right)$$

$$f_{\alpha}(\mathbf{t}) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad \alpha \in \mathbb{N}, \quad \mathbf{t} \in \mathbb{R}_+^d.$$

1990 年 10 月 10 日

1950-1951

the 1990s, the number of people in the world who are illiterate has increased from 400 million to 500 million. The number of illiterate people in the world is expected to increase to 600 million by the year 2015. The number of illiterate people in the world is expected to increase to 700 million by the year 2020. The number of illiterate people in the world is expected to increase to 800 million by the year 2025. The number of illiterate people in the world is expected to increase to 900 million by the year 2030. The number of illiterate people in the world is expected to increase to 1 billion by the year 2035. The number of illiterate people in the world is expected to increase to 1.1 billion by the year 2040. The number of illiterate people in the world is expected to increase to 1.2 billion by the year 2045. The number of illiterate people in the world is expected to increase to 1.3 billion by the year 2050. The number of illiterate people in the world is expected to increase to 1.4 billion by the year 2055. The number of illiterate people in the world is expected to increase to 1.5 billion by the year 2060. The number of illiterate people in the world is expected to increase to 1.6 billion by the year 2065. The number of illiterate people in the world is expected to increase to 1.7 billion by the year 2070. The number of illiterate people in the world is expected to increase to 1.8 billion by the year 2075. The number of illiterate people in the world is expected to increase to 1.9 billion by the year 2080. The number of illiterate people in the world is expected to increase to 2 billion by the year 2085. The number of illiterate people in the world is expected to increase to 2.1 billion by the year 2090. The number of illiterate people in the world is expected to increase to 2.2 billion by the year 2095. The number of illiterate people in the world is expected to increase to 2.3 billion by the year 2100.

[illegible]

Answers for Part B.

1. $(x - y)(x^2 + xy + y^2)$
2. $(a - 2b)(a^2 + 2ab + 4b^2)$
3. $(1 - z)(1 + z + z^2)$
4. $(1 + z)(1 - z + z^2)$
5. $(1 + 3z)(1 - 3z + 9z^2)$
6. $(2k - 3)(4k^2 + 6k + 9)$
7. $(x - 1)(x^4 + x^3 + x^2 + x + 1)$
8. $(y + 2)(y^4 - 2y^3 + 4y^2 - 8y + 16)$
9. $(2a - b)(16a^4 + 8a^3b + 4a^2b^2 + 2ab^3 + b^4)$
10. $(x - y)(x + y)(x^2 + xy + y^2)(x^2 - xy + y^2)$
11. $z^6(z - 1)(z + 1)(z^2 + z + 1)(z^2 - z + 1)$
12. $(4a - 3b)(16a^2 + 12ab + 9b^2)$
13. $\left(\frac{4}{x} - 1\right)\left(\frac{16}{x^2} + \frac{4}{x} + 1\right)$
14. $\left(\frac{10}{xy} - \frac{z}{3}\right)\left(\frac{10}{xy} + \frac{z}{3}\right)\left(\frac{100}{x^2y^2} + \frac{z^2}{9}\right)$
15. $\frac{(x^2 + 1)(x^4 - x^2 + 1)}{x^3} \left[\text{or: } \left(\frac{1}{x} + x\right)\left(\frac{1}{x^2} - 1 + x^2\right) \right]$
16. $(x - y)(x + y + 2)[(x + 1)^2 + (y + 1)^2]$
17. $(5 - 4y^2)(5 + 4y^2)$
18. $(a + b)(a^2 - ab + b^2)$
18. $(a^2 - z)(a^8 + a^6z + a^4z^2 + a^2z^3 + z^4)$
20. $(x^2 - 8)(x^2 + 8)$
21. $(a + b - 9)(a + b + 9)$
- ☆22. $(a - b - 1)([a - b]^2 + [a - b] + 1)$

- (a) $m + n = 30$ and $m - n = 2$ $[m = 16, n = 14]$
 (b) $m + n = 10$ and $m - n = 6$ $[m = 8, n = 2].]$

3. (4, 3) [Since $m^3 - n^3 = (m - n)(m^2 + mn + n^2) = 37$ and m and n are positive integers, it follows that $m^2 + mn + n^2 = 37$ and $m - n = 1$. So, $(n + 1)^2 + (n + 1)n + n^2 = 37$ and it follows that $(n + 4)(n - 3) = 0$. Hence, $n = 3$ and $m = 3 + 1$.]
4. (3, 1) [Since $2^{2i} - 3^{2j} = (2^i - 3^j)(2^i + 3^j)$, it follows that $2^{2i} - 3^{2j} = 55$ if and only if $(2^i - 3^j = 1$ and $2^i + 3^j = 55)$ or $(2^i - 3^j = 5$ and $2^i + 3^j = 11)$. So, either $2^i = 28$ and $3^j = 27$ or $2^i = 8$ and $3^j = 3$. Since there is no integer i such that $2^i = 28$, and since $2^3 = 8$ and $3^1 = 3$, the only solution is (3, 1).]
5. (2, 5), (5, 2) [Since 133 is not even, there is obviously no solution for which $m = n$. By symmetry, it is sufficient to find those solutions for which $m > n$. Since $m^2 - mn + n^2 = m(m - n) + n^2$ and m and n are positive integers, it follows that, for $m > n$, $m^2 - mn + n^2 \geq m + n$. So, since $(m + n)(m^2 - mn + n^2) = 7 \cdot 19$, either $m + n = 1$ and $m^2 - mn + n^2 = 133$ or $m + n = 7$ and $m^2 - mn + n^2 = 19$. Since $(m + n)^2 = m^2 + 2mn + n^2$ and m and n are positive integers, $(m + n)^2 > m^2 - mn + n^2$. So, $m + n \neq 1$ and [from the second possibility] $3mn = 30$. Hence, $(m - n)^2 = 9$, and $m - n = 3$. Since $m + n = 7$, the only solution [for $m > n$] is (5, 2). The only other solution is (2, 5).]
6. 3 [$m^3 - 2m^2 + 4m - 8 = (m^2 + 4)(m - 2) = 13$ if and only if $m^2 + 4 = 13$ and $m - 2 = 1$. (Since $m^2 - m + 6 = (m - \frac{1}{2})^2 + \frac{23}{4}$, $m^2 + 4 > m - 2$.) Hence, the only solution is 3.]

1. The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \int_0^x f(t) dt + \int_0^x f(t) dt + \dots$$

It is shown that the function $f(x)$ is continuous and differentiable.

2. The second part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

It is shown that the function $f(x)$ is continuous and differentiable.

3. The third part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

It is shown that the function $f(x)$ is continuous and differentiable.

4. The fourth part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

It is shown that the function $f(x)$ is continuous and differentiable.

5. The fifth part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

It is shown that the function $f(x)$ is continuous and differentiable.

6. The sixth part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

It is shown that the function $f(x)$ is continuous and differentiable.

7. The seventh part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

It is shown that the function $f(x)$ is continuous and differentiable.

8. The eighth part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

It is shown that the function $f(x)$ is continuous and differentiable.

9. The ninth part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

It is shown that the function $f(x)$ is continuous and differentiable.

$$\begin{aligned}
(m^2)^n - (m^2 - 1)n - 1 &= (m^2 - 1) \sum_{p=1}^n (m^2)^{n-p} - (m^2 - 1) \sum_{p=1}^n 1 \\
&= (m^2 - 1) \sum_{p=1}^n [(m^2)^{n-p} - 1] \\
&= (m^2 - 1) \sum_{p=1}^{n-1} [(m^2)^{n-p} - 1] \quad [(m^2)^{n-n} - 1 = 0].
\end{aligned}$$

Since, as shown above, for $m > 1$ and for each p such that $n - p \geq 1$,

$m^2 - 1 \mid (m^2)^{n-p} - 1$, it follows that $m^2 - 1 \mid \sum_{p=1}^{n-1} [(m^2)^{n-p} - 1]$. Hence,

$$(m^2 - 1)^2 \mid (m^2)^n - (m^2 - 1)n - 1.$$

*

Answers for Part E.

Some solutions to these equations may be found by inspecting a table of squares and cubes. For example, it is easy in the case of the Sample to see that $49 - 36 = 13$. So, $(7, 6)$ is a solution. However, we need to be sure that there are no other solutions. The factoring procedure gives this assurance.

1. $(10, 9)$

2. $(8, 2), (16, 14)$ [Since $m^2 - n^2 = (m + n)(m - n)$ and m and n are positive integers, $m + n > m - n$. Since $(m + n)(m - n) = 60$ --an even number--one of the numbers $m + n$ and $m - n$ must be even. Since m and n are integers, it follows that both must be even. So, the only possibilities are

2^{q-p} is an integer. By Theorem 150c, $(-1)^{p-1}$ is an integer. Hence, by Theorem 110d, for $q \geq p$, $2^{q-p}(-1)^{p-1}$ is an integer. Using Theorem 110b it is now easy to prove by induction that

$\sum_{p=1}^q 2^{q-p}(-1)^{p-1}$ is an integer. Hence, $3 \mid 2^q - (-1)^q$.

$$\begin{aligned}
 2. \text{ By Exercise 1, } \frac{2^q - (-1)^q}{3} &= \sum_{p=1}^q 2^{q-p}(-1)^{p-1} \\
 &= \sum_{p=1}^{q-1} 2^{q-p}(-1)^{p-1} + (-1)^{q-1} \\
 &= 2 \sum_{p=1}^{q-1} 2^{q-p-1}(-1)^{p-1} + (-1)^{q-1}.
 \end{aligned}$$

As in Exercise 1, $\sum_{p=1}^{q-1} 2^{q-1-p}(-1)^{p-1}$ is an integer. Since $(-1)^{q-1} = 1$ or -1 , it follows that $[2^q - (-1)^q]/3$ is odd.

★3. [See answer for Exercise ★4, below ($m = 2$).]

★4. By Theorem 170, $(m^2)^q - 1 = (m^2 - 1) \sum_{p=1}^q (m^2)^{q-p}$ and, as in Exercise 1, it follows that, for $m > 1$ and for each q , $m^2 - 1 \mid (m^2)^q - 1$. Similarly,

Answers for Part C.

1. $6x^3(2 - 5xy)$

2. $8ab^2(ab - 3a + b)$

3. $(m + q)(m + p)$

4. $(t + 5)(t - s)$

5. $(z^3 + 2)(2z - 1)$

6. $(x - y)(3p - q)$

7. $(x - 3)(x + 3)(y^2 - 6)$

8. $(t - z)(t + 3)(k - 4)(k + 4)$

9. $(b + 1)(b^2 - b + 1)(3a - 1)(3a + 1)$

10. $(x - 1)(x + 1)(x^2 - x + 1)(x^4 + x^3 + x^2 + x + 1)$

11. $(2a + 5b)(2a - 5b + 1)$

12. $4a^3b(a + 3)(7a - 5)$

13. $(a^2 - 4a + 16)(a^2 + 4a + 16)$

14. $(x^2 - 2xy - y^2)(x^2 + 2xy - y^2)$

15. $(a^2 - 3ab + b^2)(a^2 + 3ab + b^2)$

16. $(x^2 - 2xy + 2y^2)(x^2 + 2xy + 2y^2)$

17. $(x - 2)(x^2 + 4)$

18. $(x - 2)(x - 1)(x + 1)(x + 2)$

$$[x^4 - 5x^2 + 4 = (x^4 - 4x^2 + 4) - x^2 = (x^2 - 2)^2 - x^2 = \dots;$$

$$x^4 - 5x^2 + 4 = (x^2 - 4)(x^2 - 1) = \dots]$$

19. $7(2 + u)(4 - 2u + u^2)$

20. $(t - w)(t + w)^2$

*

Answers for Part D.

1. By Theorem 170, $2^q - (-1)^q = (2 - -1) \sum_{p=1}^q 2^{q-p}(-1)^{p-1}$
 $= 3 \sum_{p=1}^q 2^{q-p}(-1)^{p-1}$. It follows from Theorem 151a that [for $q \geq p$]

★7. John's date is Clara, Henry's is Mary, and Bill's is Kate.

[Let J , H , M , and K be the average scores (and numbers of games played) by John, Henry, Mary, and Kate, and let j and h be the average scores (and numbers of games played) by John's date and Henry's date, respectively. Then, the total number of points which, for example, John made is J^2 . The conditions of the problem are that $J^2 - j^2 = 63 = H^2 - h^2$, that $J - M = 23$, and that $H - K = 11$.

Since $(J - j)(J + j) = 63$ and J and j are positive integers, it follows that either $J - j = 1$ and $J + j = 63$, or $J - j = 3$ and $J + j = 21$, or $J - j = 7$ and $J + j = 9$. In these cases, (J, j) is $(32, 31)$, $(12, 9)$, or $(8, 1)$, respectively. Similarly, (H, h) is $(32, 31)$, $(12, 9)$, or $(8, 1)$. Since $J - M = 23$, $J \geq 24$. So, $J = 32$, $j = 31$, and $M = 9$. Already, since $j \neq M$, we see that John's date was not Mary. Since $H - K = 11$, $H \geq 12$. So, (H, h, K) is $(32, 31, 21)$ or $(12, 9, 1)$. In either case, $h \neq K$ and, since $j = 31$, $j \neq K$. So, Kate was neither Henry's nor John's date. Hence, Kate was Bill's date and, since John's date was neither Mary nor Kate, John's date was Clara. Finally, by elimination, Henry's date was Mary.]

*

Answers for Part F.

- | | | |
|---|------------------------------|----------------------|
| 1. $(x + 3y - t)(x + 3y + t)$ | 2. $y^2(y - 1)$ | 3. $(b - 17)(b + 5)$ |
| 4. $(2z - 1)(z + 8)$ | 5. $(pq - 9)(pq + 9)$ | 6. $5s(6r - 5s)$ |
| 7. $(x - 2^{-1})(x + 2^{-1})(x^2 + 2^{-2})(x^4 + 2^{-4})$ | | |
| 8. $(1 - a + b)(1 + a - b + [a - b]^2)$ | | |
| 9. $(a + b - c)(a + b + c)$ | 10. $(xy - 8)(xy + 3)$ | |
| 11. $(5 + s)(5^2 - 5s + s^2)$ | 12. $(t - s - a)(t - s + a)$ | |

13. $(2a^2 - 9ay - 3y^2)(2a^2 + 9ay - 3y^2)$

14. $(3a + 2)(a + 1)$

16. $(t^2 - 12r^2)(t^2 + 11r^2)$

18. $(x - 4a - 3b)(x - 4a + 3b)$

20. $(a - 1)(a + 1)(a^2 + 1)(a^4 + 1)$

22. $(u^2 + v^2 - x^2 - y^2)(u^2 + v^2 + x^2 + y^2)$

24. $(u - v - a + b)(u - v + a - b)$

26. $(1 - 3y)(1 + 3y + 9y^2)$

28. $(x + y - z)(x + y + z)$

30. $(5 - 6y)(5 + 2y)$

32. $(x + 2c - 1)(x + 2c + 1)$

15. $(5 - s)(2 - 15s)$

17. $(1 - x + y)(1 + x - y)$

19. $(a^2 + 2b^2c^2)^2$

21. $(t + 12)(t + 8)$

23. $(sx - ty)(2x + 3y)$

25. $(4 + m)(16 - 4m + m^2)$

27. $(3m + 1)(9m^2 - 3m + 1)$

29. $(y^3 - 5)(y^3 + 5)$

31. $(b + y - x)(b + y + x)$

33. $(x + 5)(17x - 1)$

*

Answers for Part G.

1. $x + y$

2. $p^p + r^q$

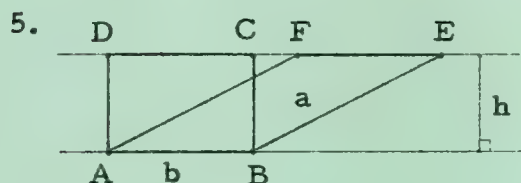
3. $\frac{y}{x}$

4. $\frac{c(a + c)}{a}$

Answers for Miscellaneous Exercises.

[Easy: 2-6; Medium: 1, 8; Hard: 7]

1. (a) 1 (b) $\frac{(a+b)(a-b)}{2(a^2+b^2)}$
2. (a) 15 (b) $1/2$ (c) $-4\sqrt{5}$ (d) $8/11$
3. $9\frac{1}{11}$ hours 4. $\pi ab(b+2c)$



$$a \geq h \Rightarrow 2(a+b) \geq 2(h+b)$$

6. (a) $(x+12)(x-3)$ (b) $(a-13)(a+9)$ (c) $(11+x)(10-x)$

7. Guess: $\forall_n \sum_{p=1}^{2n-1} [p + (n-1)^2] = n^3 + (n-1)^3$

Proof: $\sum_{p=1}^{2q-1} [p + (q-1)^2] = \frac{(2q-1)2q}{2} + (2q-1)(q-1)^2$

$$= (2q-1)[q + (q-1)^2];$$

$$q^3 + (q-1)^3 = [q + (q-1)][q^2 - q(q-1) + (q-1)^2]$$

$$= (2q-1)[q + (q-1)^2]$$

8. If $\frac{a}{b} = \frac{b}{c}$ then $\frac{b}{c} = \frac{a+b}{b+c}$. So, $\frac{a}{b} \cdot \frac{b}{c} = \frac{a+b}{b+c} \cdot \frac{a+b}{b+c}$.

*

line 8b on page 8-166: Although the components of $\{\{b, c, f\}, \{c, d, e, f\}\}$ are, respectively, a 3-membered and a 4-membered subset of S, the first component is not a subset of the second.

line 2b: A 4-membered set has 4 3-membered subsets, each of which is obtained by discarding one of the 4 members of the given set.

4. $C(9, 6)$ [or: 84] [Alternatively, the number in question is the total number of 7-membered subsets of S minus the number of those which do not contain the number 6--that is, it is $C(10, 7) - C(9, 7)$. By (*) on page 8-166, $C(10, 7) - C(9, 7) = C(9, 6)$. Compare this with the reasoning on page 7-72 of Unit 7 which originally led to the discovery of (*).]
5. $C(5, 3)$ [or: 10]
6. $C(10, 3) - [C(5, 3) + C(5, 3)]$ [or: 100] [The number in question is the total number of 3-membered subsets of S minus the number of those which consist exclusively of even numbers or exclusively of odd numbers.]

*

In connection with the exercises of Part B and later counting exercises, you may wish to introduce the word 'probability'. Before doing so, re-read the COMMENTARY for page 4-G of Unit 4. You may also find interesting Chapter 3 of Probability: A First Course by Mosteller, Rourke and Thomas, [Reading, Mass: Addison-Wesley, 1961]. Examples of questions you might ask in connection with Part B are:

What is the probability that a 7-membered subset of S contains the number 6?

[Answer (from Exercises 1 and 4 of Part B): $C(9, 6)/C(10, 7)$,
or: 0.7]

What is the probability that a 7-membered subset of S does not contain the number 6?

[Answer: $C(9, 7)/C(10, 7)$, or: 0.3, or (from the previous question) $1 - 0.7$]

What is the probability that a 3-membered subset of S contains only even numbers?

[Answer: $C(5, 3)/C(10, 3)$, or: $1/12$]

What is the probability that a 7-membered subset of S contains only even numbers?

[Answer: $C(5, 7)/C(10, 7)$, or: 0]

$$\begin{aligned}
2. \quad C(9, 6) &= C(9, 5) \cdot \frac{9-5}{6} \\
&= C(9, 4) \cdot \frac{9-4}{5} \cdot \frac{4}{6} \\
&= C(9, 3) \cdot \frac{9-3}{4} \cdot \frac{5}{5} \cdot \frac{4}{6} \\
&= C(9, 2) \cdot \frac{9-2}{3} \cdot \frac{6}{4} \cdot \frac{5}{5} \cdot \frac{4}{6} \\
&= C(9, 1) \cdot \frac{9-1}{2} \cdot \frac{7}{3} \cdot \frac{6}{4} \cdot \frac{5}{5} \cdot \frac{4}{6} \\
&= C(9, 0) \cdot \frac{9-0}{1} \cdot \frac{8}{2} \cdot \frac{7}{3} \cdot \frac{6}{4} \cdot \frac{5}{5} \cdot \frac{4}{6} \\
&= 1 \cdot \frac{9}{1} \cdot \frac{8}{2} \cdot \frac{7}{3} \cdot \frac{6}{4} \cdot \frac{5}{5} \cdot \frac{4}{6} = 84
\end{aligned}$$

$$3. \quad C(10, 7) = 1 \cdot \frac{10}{1} \cdot \frac{9}{2} \cdot \frac{8}{3} \cdot \frac{7}{4} \cdot \frac{6}{5} \cdot \frac{5}{6} \cdot \frac{4}{7} = \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = 120$$

$$4. \quad C(3, 5) = C(3, 4) \cdot \frac{3-4}{5} = C(3, 3) \cdot \frac{3-3}{4} \cdot \frac{3-4}{5} = 0$$

*

Answers for Part B.

[The principal value of these exercises lies in the practice they give in expressing answers to counting problems in terms of values of C . The computations of the relevant values have, for the most part, been carried out in working the exercises of Part A.] [Note that Exercise 6 is on page 8-169.]

1. $C(10, 7)$ [or: 120]

2. $C(10, 3)$ [Since to each 7-membered subset there corresponds exactly one 3-membered subset (its complement), and conversely, $C(10, 3) = C(10, 7)$. So, an alternative answer is '120'.]

3. $C(9, 7)$ [$C(9, 7) = C(9, 6) \cdot \frac{9-6}{7} = 84 \cdot \frac{3}{7} = 36$]

line 7 on page 8-167: The third entry in the second column is incorrect since $\{a, b, d\} \not\subseteq \{a, b, e, f\}$

line 11: A given 3-membered set can be enlarged to a 4-membered set by adjoining one additional member. Since S has 6 members, there are $9 - 3$ choices for the additional member.

line 18: See (**) at the bottom of the page.

line 24: Generalizing the explanation for line 2b on page 8-166, a given $(k + 1)$ -membered set has $k + 1$ k -membered subsets, each of which is obtained by discarding one of the $k + 1$ members of the given set.

line 27: Generalizing the explanation for line 11, above, a given k -membered subset of a j -membered set can be enlarged to a $(k + 1)$ -membered subset by adjoining any one of the $j - k$ members not already belonging to it.

*

As illustrated by the computation on page 8-168, use of the new recursive definition of C involves much less labor than does use of the old definition given in connection with (*) on page 8-166. [To compute $C(12, 5)$ by the latter method requires the preliminary computation of 56 values of C.] Moreover, the new formula leads directly to the explicit definition of C [Theorem 171 on page 8-169] and the useful Theorem 172 on page 8-171.

*

Answers for Part A.

$$\begin{aligned} 1. \quad C(5, 3) &= C(5, 2) \cdot \frac{5-2}{3} \\ &= C(5, 1) \cdot \frac{5-1}{2} \cdot \frac{3}{3} \\ &= C(5, 0) \cdot \frac{5-0}{1} \cdot \frac{4}{2} \cdot \frac{3}{3} \\ &= 1 \cdot \frac{5}{1} \cdot \frac{4}{2} \cdot \frac{3}{3} = 10 \end{aligned}$$

1. The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{2\pi n x}{1+x^2}.$$

It is shown that $f(x)$ is a continuous function of x and that it satisfies the equation

$$f(x) = \frac{1}{2} \left(f(x+1) + f(x-1) \right) + \frac{1}{2} \left(f(x+i) + f(x-i) \right).$$

$$f(x) = \frac{1}{2} \left(f(x+1) + f(x-1) \right) + \frac{1}{2} \left(f(x+i) + f(x-i) \right).$$

It is

shown that

$$f(x) = \frac{1}{2} \left(f(x+1) + f(x-1) \right) + \frac{1}{2} \left(f(x+i) + f(x-i) \right).$$

Answers for Part E.

$$1. \quad 6! = 5! \cdot 6 = 4! \cdot 5 \cdot 6 = 3! \cdot 4 \cdot 5 \cdot 6 = 2! \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 1! \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \\ = 0! \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 1 \cdot 720 = 720$$

$$2. \quad 1$$

$$3. \quad 30$$

$$4. \quad 120$$

$$5. \quad 252$$

$$6. \quad 9366819 \text{ [or: } 3 \cdot 7 \cdot 11 \cdot 23 \cdot 41 \cdot 43]$$

[In computations such as that of Exercise 6, a factored answer is usually sufficient, and often more convenient. Here is a typical computation:

$$\frac{46!}{40!6!} = \frac{\overset{23}{\cancel{46}} \cdot \overset{3}{\cancel{45}} \cdot \overset{11}{\cancel{44}} \cdot 43 \cdot \overset{7}{\cancel{42}} \cdot 41}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot \cancel{6}} = 3 \cdot 7 \cdot 11 \cdot 23 \cdot 41 \cdot 43]$$

*

Answers for Part F.

$$1. \quad 7!/4!$$

$$2. \quad 19!/15!$$

$$3. \quad 52!/47!$$

The answer for Exercise 6 of Part B is in the COMMENTARY for pages 8-167 and 8-168.

*

Answers for Part C.

1. For $j \geq 0$, $C(j, 1) = C(j, 0) \cdot \frac{j - 0}{0 + 1} = 1 \cdot j = j$.

2. For $j \geq 0$, $C(j, 2) = C(j, 1) \cdot \frac{j - 1}{1 + 1} = \frac{j(j - 1)}{2}$.

*

Answers for Part D.

1. (i) By Exercise 1 of Part C, $C(0, 1) = 0$.

(ii) Suppose that $C(0, p) = 0$. Since $C(0, p + 1) = C(0, p) \cdot \frac{0 - p}{p + 1}$, it follows that $C(0, p + 1) = 0$. Hence,

$$\forall_n [C(0, n) = 0 \Rightarrow C(0, n + 1) = 0].$$

(iii) From (i) and (ii) it follows, by the PMI, that $\forall_n C(0, n) = 0$.

☆2. (i) For $j \geq 0$, $C(j, j + 1) = C(j, j) \cdot \frac{j - j}{j + 1} = C(j, j) \cdot 0 = 0$.

(ii) Suppose, for a $j \geq 0$ and a $k \geq j + 1$, that $C(j, k) = 0$. Since $C(j, k + 1) = C(j, k) \cdot \frac{j - k}{k + 1}$, it follows that $C(j, k + 1) = 0$.

Hence, for $j \geq 0$, $\forall_{k \geq j + 1} [C(j, k) = 0 \Rightarrow C(j, k + 1) = 0]$.

(iii) From (i) and (ii) it follows, by Theorem 114, that, for $j \geq 0$,

$$\forall_{k \geq j + 1} C(j, k) = 0.$$

Consequently, $\forall_{j \geq 0} \forall_{k \geq j + 1} C(j, k) = 0$.

*

100

10

100

•

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega \quad (1)$$

Chlorophyll $\frac{0.7}{0.8}$

*

Answers for Part H.

1. 210 [A short cut: $C(10, 6) = C(10, 4) = \frac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4}$]
2. 1001
3. 177100 [or: $2^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 23$]
4. 2598960 [or: $2^4 \cdot 3 \cdot 5 \cdot 7^2 \cdot 13 \cdot 17$]
5. 120
6. 7128
7. 1
8. 198

*

Answer for Part ★I.

$$\begin{aligned}
 C(p-1, q) + C(p-1, q-1) &= \frac{\prod_{i=0}^{q-1} (p-1-i)}{q!} + \frac{\prod_{i=0}^{q-2} (p-1-i)}{(q-1)!} \\
 \text{Th. 171} \nearrow & \\
 &= \frac{\prod_{i=0}^{q-2} (p-1-i) \cdot (p-q)}{q!} + \frac{\prod_{i=0}^{q-2} (p-1-i) \cdot q}{(q-1)! \cdot q} \\
 &= \frac{\prod_{i=0}^{q-2} (p-1-i) \cdot p}{q!} = \frac{p \cdot \prod_{i=1}^{q-1} (p-i)}{q!} \\
 &\quad \text{Th. 148} \nearrow \\
 \text{Th. 147} \rightarrow &= \frac{\prod_{i=0}^{q-1} (p-i)}{q!} = C(p, q) \\
 &\quad \text{Th. 171} \nearrow
 \end{aligned}$$

[For $q \leq p-1$, factorial notation can be used throughout:

$$\begin{aligned}
 C(p-1, q) + C(p-1, q-1) &= \frac{(p-1)!}{q! (p-1-q)!} + \frac{(p-1)!}{(q-1)! (p-q)!} \\
 &= \frac{(p-1)! \cdot (p-q)}{q! (p-1-q)! (p-q)} + \frac{(p-1)! \cdot q}{(q-1)! (p-q)! q} \\
 &= \frac{(p-1)! \cdot p}{q! (p-q)!} = \frac{p!}{q! (p-q)!} = C(p, q) \quad]
 \end{aligned}$$

One consequence of Theorem 172 is that, for $j \geq 0$ and $k \geq 0$, $\frac{(j+k)!}{j!k!}$ is an integer [for, $C(j+k, k)$ is a nonnegative integer]. It may interest some of your students to see how this can be established without using Theorem 172. Now, if either $j = 0$ or $k = 0$, $\frac{(j+k)!}{j!k!} = 1$. So, all that needs to be shown is that, for any m and n , and any prime number p , the sum of the exponents of the largest powers of p which divide $m!$ and $n!$, respectively, is less than or equal to the exponent of the largest power of p which divides $(m+n)!$. As pointed out in the COMMENTARY for Exercise 36 on page 8-156, these three exponents are, respectively,

$$\llbracket m/p \rrbracket + \llbracket m/p^2 \rrbracket + \llbracket m/p^3 \rrbracket + \dots,$$

$$\llbracket n/p \rrbracket + \llbracket n/p^2 \rrbracket + \llbracket n/p^3 \rrbracket + \dots,$$

and $\llbracket (m+n)/p \rrbracket + \llbracket (m+n)/p^2 \rrbracket + \llbracket (m+n)/p^3 \rrbracket + \dots,$

Now, as proved in Exercise 3(h) of Part A on page 7-103 of Unit 7,

$$\forall_x \forall_y \llbracket x \rrbracket + \llbracket y \rrbracket \leq \llbracket x+y \rrbracket.$$

Hence, for each q ,

$$\llbracket m/p^q \rrbracket + \llbracket n/p^q \rrbracket \leq \llbracket (m+n)/p^q \rrbracket$$

and so, for each p , the sum of the first two exponents is less than or equal to the third.

An entirely similar argument shows, for example, that if $m = n_1 + n_2 + n_3$ then $\frac{m!}{n_1!n_2!n_3!}$ is an integer.

*

That a j -membered set has [for $0 \leq k \leq j$] the same number of k -membered subsets as $(j-k)$ -membered subsets, is obvious when one notices that the complement of each k -membered subset of a j -membered set is a $(j-k)$ -membered set, and vice versa. [See the corollary of Theorem 172 on page 8-172.]

*

Answers for Part G.

$$1. \frac{35!}{32!3!} \qquad 2. \frac{176!}{85!91!} \qquad 3. \frac{25!}{5!20!} \qquad 4. \frac{25!}{20!5!}$$

1. The first part of the report is a general
 introduction to the subject of the study.
 2. The second part is a detailed description of the
 methods used in the study.

3. The third part is a discussion of the results of the study.
 4. The fourth part is a conclusion and a list of references.

5. The fifth part is a list of the names of the
 authors of the study.
 6. The sixth part is a list of the names of the
 institutions where the study was conducted.
 7. The seventh part is a list of the names of the
 people who assisted in the study.
 8. The eighth part is a list of the names of the
 people who reviewed the study.

9. The ninth part is a list of the names of the
 people who were interviewed for the study.
 10. The tenth part is a list of the names of the
 people who were observed for the study.
 11. The eleventh part is a list of the names of the
 people who were asked to participate in the study.
 12. The twelfth part is a list of the names of the
 people who were asked to provide information for the study.

One consequence of Theorem 172 is that, for $j \geq 0$ and $k \geq 0$, $\frac{(j+k)!}{j!k!}$ is an integer [for, $C(j+k, k)$ is a nonnegative integer]. It may interest some of your students to see how this can be established without using Theorem 172. Now, if either $j = 0$ or $k = 0$, $\frac{(j+k)!}{j!k!} = 1$. So, all that needs to be shown is that, for any m and n , and any prime number p , the sum of the exponents of the largest powers of p which divide $m!$ and $n!$, respectively, is less than or equal to the exponent of the largest power of p which divides $(m+n)!$. As pointed out in the COMMENTARY for Exercise 36 on page 8-156, these three exponents are, respectively,

$$\llbracket m/p \rrbracket + \llbracket m/p^2 \rrbracket + \llbracket m/p^3 \rrbracket + \dots,$$

$$\llbracket n/p \rrbracket + \llbracket n/p^2 \rrbracket + \llbracket n/p^3 \rrbracket + \dots,$$

and $\llbracket (m+n)/p \rrbracket + \llbracket (m+n)/p^2 \rrbracket + \llbracket (m+n)/p^3 \rrbracket + \dots,$

Now, as proved in Exercise 3(h) of Part A on page 7-103 of Unit 7,

$$\forall_x \forall_y \llbracket x \rrbracket + \llbracket y \rrbracket \leq \llbracket x+y \rrbracket.$$

Hence, for each q ,

$$\llbracket m/p^q \rrbracket + \llbracket n/p^q \rrbracket \leq \llbracket (m+n)/p^q \rrbracket$$

and so, for each p , the sum of the first two exponents is less than or equal to the third.

An entirely similar argument shows, for example, that if $m = n_1 + n_2 + n_3$ then $\frac{m!}{n_1!n_2!n_3!}$ is an integer.

*

That a j -membered set has [for $0 \leq k \leq j$] the same number of k -membered subsets as $(j-k)$ -membered subsets, is obvious when one notices that the complement of each k -membered subset of a j -membered set is a $(j-k)$ -membered set, and vice versa. [See the corollary of Theorem 172 on page 8-172.]

*

Answers for Part G.

1. $\frac{35!}{32!3!}$

2. $\frac{176!}{85!91!}$

3. $\frac{25!}{5!20!}$

4. $\frac{25!}{20!5!}$

TC[8-171, 172]a

1. The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \sum_{n=0}^{\infty} a_n x^n$, where a_n are the coefficients of the power series. It is shown that $f(x)$ is a continuous function of x and that it satisfies the functional equation $f(x) = f(x^2) + x$.

2. In the second part, we consider the problem of the representation of the function $f(x)$ as a sum of two functions, one of which is a polynomial and the other is a function which is continuous and satisfies the same functional equation as $f(x)$. It is shown that such a representation is possible and that the polynomial is unique.

3. The third part of the paper is devoted to the study of the properties of the function $g(x)$ defined by the equation $g(x) = \sum_{n=0}^{\infty} b_n x^n$, where b_n are the coefficients of the power series. It is shown that $g(x)$ is a continuous function of x and that it satisfies the functional equation $g(x) = g(x^2) + x^2$.

4. In the fourth part, we consider the problem of the representation of the function $g(x)$ as a sum of two functions, one of which is a polynomial and the other is a function which is continuous and satisfies the same functional equation as $g(x)$. It is shown that such a representation is possible and that the polynomial is unique.

5. The fifth part of the paper is devoted to the study of the properties of the function $h(x)$ defined by the equation $h(x) = \sum_{n=0}^{\infty} c_n x^n$, where c_n are the coefficients of the power series. It is shown that $h(x)$ is a continuous function of x and that it satisfies the functional equation $h(x) = h(x^2) + x^3$.

6. In the sixth part, we consider the problem of the representation of the function $h(x)$ as a sum of two functions, one of which is a polynomial and the other is a function which is continuous and satisfies the same functional equation as $h(x)$. It is shown that such a representation is possible and that the polynomial is unique.

$N(B) - N(A \cap B) = 0$. This is an example of two wrongs making a right. If one avoids the first error and assumes, correctly, that, since $N(A) = N(B)$ and A is infinite, $N(A) + N(B) = N(A)$, he obtains ' $N(A \cup B) = N(A) - N(A \cap B)$ '. Repeating the second error, he obtains an obviously false result-- $N(A \cup B) = 0$.)]

lines 7b and 6b: We give below a derivation of (***) from (**) and various theorems about sets (and about subtraction of real numbers). The theorems about sets can be derived from the basic principles given in Unit 5.

$$A \cup B = A \cup (\tilde{A} \cap B), \quad \text{and} \quad A \cap (\tilde{A} \cap B) = \emptyset.$$

Hence, by (**), $N(A \cup B) = N(A) + N(\tilde{A} \cap B)$.

$$B = (\tilde{A} \cap B) \cup (A \cap B), \quad \text{and} \quad (\tilde{A} \cap B) \cap (A \cap B) = \emptyset.$$

Hence, by (**), $N(B) = N(\tilde{A} \cap B) + N(A \cap B)$,

and, so, $N(\tilde{A} \cap B) = N(B) - N(A \cap B)$. Consequently,

$$\begin{aligned} N(A \cup B) &= N(A) + [N(B) - N(A \cap B)] \\ &= N(A) + N(B) - N(A \cap B). \end{aligned}$$

Solution of problem at foot of page 8-174:

If A is the set of those families each of which has an automobile, and T is the set of those each of which has a television set, then $N(A \cup T) = 10$, $N(A) = 8$, and $N(T) = 9$. Since, by (***), $N(A \cap T) = N(A) + N(T) - N(A \cup T)$, $N(A \cap T) = 7$. So, there are 7 families which have both an automobile and a television set. Since 8 families have automobiles, and only 7 of these also have television sets, just 1 family has an automobile but no television set.

line 13: 2502 [The committees which contain at least 3 seniors are
those which do not contain at most 2 seniors,
 $C(15, 5) - 501 = 3003 - 501]$

line 19: $A \times B$ is the union of disjoint sets, each of which consists of
all members of $A \times B$ which have a given first component.
Since there is a one-to-one correspondence between these sets
and the members of A , there are, by (C_1) , $N(A)$ such sets. For
each of these sets, there is a one-to-one correspondence be-
tween its members and those of B . So, by (C_1) , each set has
 $N(B)$ members. Hence, by (C_2) , $N(A \times B) = \sum_{i=1}^{N(A)} N(B)$.

line 23: Students have probably been thinking only of finite sets. How-
ever, infinite sets have cardinal numbers [although these
cannot, as in the case of finite sets, be correlated with the
nonnegative integers] and, while (C_2) and $(*)$ hold for any sets,
finite or infinite, $(***)$ does not. Actually, $(***)$ does hold
whenever either A or B is finite. It fails to hold just in case
 A , B , and $A \cap B$ are all infinite and $N(A) = N(B) = N(A \cap B)$.
In this case, $N(A) + N(B) = N(A)$ and, because $N(A \cap B) = N(A)$,
' $N(A) - N(A \cap B)$ ' is not defined. [For reasons which will not
be gone into here, subtraction of infinite cardinals can justifi-
ably be defined only when the subtrahend is less than the
minuend--for a cardinal number c , $c - c = 0$ only if c is finite.
Amusingly enough, even in the case when $(***)$ fails to hold,
uncritical simplification of its right side can lead to the correct
result, ' $N(A \cup B) = N(A)$ '. (The uncritical steps are assuming
that $N(A) + N(B) - N(A \cap B) = N(A) + [N(B) - N(A \cap B)]$ and
that, since, in the case in question, $N(B) = N(A \cap B)$,

There is a great variety of counting [or: combinatorial] problems, One's success in solving such a problem depends largely on the insight he develops by thinking hard about it. This might be said, truly, about any kind of problem for which one has not memorized an algorithm, but it applies especially to combinatorial problems because, due to their great variety, algorithms for them are hard to come by. However, in simple cases it is often possible so to analyze a counting problem as to see how to express its answer in terms of values of the function C of Theorem 171 or of the function P of Theorem 174 [on page 8-180]. For $j \geq 0$ and $k \geq 0$, $C(j, k)$ is the number of k -membered subsets of a j -membered set--in combinatorial language, the number of combinations of j things taken k at a time--, and $P(j, k)$ is the number of ordered k -membered subsets of a j -membered set--that is, the number of permutations of j things taken k at a time. [The words 'combination' and 'permutation' are introduced in the text on pages 8-178 and 8-179.] Consequently, it is customary to divide the chapter on counting problems in, say, a "college algebra" text into two parts--one labelled 'permutations', the other 'combinations'. An instructor using such a text is fortunate to find as many as three class-hours to divide between the two parts--with a fourth hour for the chapter on probability which traditionally follows the one on permutations and combinations. The results are what one might expect. Students learn fairly readily to compute values of C and P . But, their success in learning to solve counting problems is indicated by the common plaint: "I could do it if I knew whether it was a permutation or a combination." [Usually it is neither.] Their difficulties are of the same nature as those they experienced much earlier when they may have said, "I could do the problem if I knew whether to multiply or divide."

Your students' familiarity with sets and their experiences with counting problems in earlier units [see, for example, pages 4-12 ff. of Unit 4, pages 51-62 ff. of Unit 5, and pages 7-71 ff. and page 7-91 of Unit 7] should have prepared them for the pages which follow.

*

In Units 4 and 5, ' $n(A)$ ' was used for 'the number of members of A '. Here, to avoid confusion with the use of ' n ' as a variable, we use ' N ' in place of ' n '.

*

The counting principles $(*)$, (C_1) , and (C_2) are of fundamental importance. In a treatment of cardinal numbers--independent of the study of real numbers-- (C_1) serves as a criterion for the adequacy of

(b) At least 5 students take none of the three subjects.

[For this part, $N(M \cap E \cap F) < 63$. So, by (*), above, $N(M \cup E \cup F) < 33 + 63 = 96$. Since there are 100 students, more than $100 - 96$ take none of the subjects.]

(c) The data are inconsistent.

[By (*), if $N(M \cap E \cap F) = 68$ then $N(M \cup E \cup F) = 101$. But, $N(M \cup E \cup F) \leq 100$.]

[You can find other problems like Exercise 11, as well as a diagrammatic method of solution, in Introduction to Finite Mathematics, by Kemeny, Snell, and Thompson [New York: Prentice-Hall, 1957] pages 88 ff.]

12. [See discussion on page 8-177.]

the 1990s, the number of people in the world who are under 15 years of age is expected to increase by 1.5 billion, from 1.2 billion in 1990 to 2.7 billion in 2010. The number of people aged 65 and over is expected to increase by 1.2 billion, from 0.4 billion in 1990 to 1.6 billion in 2010. The number of people aged 15-64 is expected to increase by 1.5 billion, from 1.2 billion in 1990 to 2.7 billion in 2010. The number of people aged 65 and over is expected to increase by 1.2 billion, from 0.4 billion in 1990 to 1.6 billion in 2010. The number of people aged 15-64 is expected to increase by 1.5 billion, from 1.2 billion in 1990 to 2.7 billion in 2010.

[illegible]

1. *Phragmites australis* (Cav.) Trin. ex Steud.

1. *Chlorophyll a* (Chl *a*)

1. *Phragmites australis* (Cav.) Trin. ex Steud.

$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$

1972

Journal of Management Studies, 19(1), 67-80.

Journal of Management Studies, 19(1), 67-80.

Figure 1. The effect of the concentration of the *Agrobacterium* suspension on the transformation efficiency of *Agrobacterium* strains.

$$C(2n, n) = C(2n - 1, n) + C(2n - 1, n - 1)$$

and, since $n + (n - 1) = 2n - 1$, $C(2n - 1, n) = C(2n - 1, n - 1)$.]

9. 12; 10 [Let B be the set of guinea pigs with some black, W the set of guinea pigs with some white. Since

$$N(B \cup W) = N(B) + N(W) - N(B \cap W),$$

it follows from the given data that

$$39 = 29 + 22 - N(B \cap W).$$

Since the set of spotted guinea pigs is $B \cap W$, the number of such animals is 12.

Let P be the set of pure white guinea pigs. Since $P \cap (B \cap W) = \emptyset$ and $W = P \cup (B \cap W)$, it follows, by (C_2) , that $22 = N(P) + 12$.]

10. (1) $N((A \cup B) \cup C) = N(A \cup B) + N(C) - N((A \cup B) \cap C),$

(2) $N(A \cup B) = N(A) + N(B) - N(A \cap B),$ and,

since $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

and $(A \cap C) \cap (B \cap C) = A \cap B \cap C,$

(3) $N((A \cup B) \cap C) = N(A \cap C) + N(B \cap C) - N(A \cap B \cap C).$

Substituting from (2) and (3) in (1),

$$\begin{aligned} N(A \cup B \cup C) &= N(A) + N(B) + N(C) \\ &\quad - [N(A \cap B) + N(B \cap C) + N(C \cap A)] \\ &\quad + N(A \cap B \cap C). \end{aligned}$$

[This result is generalized in (C_4) on page 8-193.]

11. (a) 67 [Let M be the set of students taking math, E that of those taking English, and F that of those taking French. Using the formula of Exercise 10 and the given data,

$$\begin{aligned} (*) \quad N(M \cup E \cup F) &= 95 + 85 + 73 - [70 + 80 + 70] \\ &\quad + N(M \cap E \cap F). \end{aligned}$$

Since every student takes at least one of the three subjects, $N(M \cup E \cup F) = 100$.]

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TO THE EDITOR
OF THE JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY

RE: [Title of the paper]

Dear Sirs,

I am writing to you regarding the paper of [Author's Name] in the [Journal Name], [Volume], [Issue], [Page Numbers], [Year].

The paper is very interesting and I have read it with great interest. I have some comments on it which I would like to discuss with you. I am particularly interested in the [Specific Topic] and the [Specific Topic]. I have some questions about the [Specific Topic] and the [Specific Topic]. I would like to know if you have any answers to these questions. I would also like to know if you have any other comments on the paper.

I am sure that you will be able to help me with these questions. I would be very grateful if you could reply to me as soon as possible.

I am sure that you will be able to help me with these questions. I would be very grateful if you could reply to me as soon as possible.

Yours faithfully,
[Signature]

[Name]

cc: [Name]

4. (a) 2118760 [= $C(50, 5)$]

(b) 218246 [The number of selections with at least 2 defective articles is the total number, 2118760, of selections minus the number with at most one defective article. The latter is $C(44, 5) \cdot C(6, 0) + C(44, 4) \cdot C(6, 1)$ [= $1086008 \cdot 1 + 135751 \cdot 6$].]

5. 447051 [= $C(23, 6) + C(23, 6) + C(23, 7) = 100947 \cdot 2 + 245157$]
 with just one with neither
 of the 2 of the 2

6. 15 [= $C(6, 4)$, or: = $C(6, 2)$]; 22 [= $C(6, 0) + C(6, 1) + C(6, 2)$]

7. 65 [= $2 \cdot C(10, 9) + C(10, 8)$, or: = $C(12, 10) - C(10, 10)$]

8. (a) 35 [= $C(7, 3)$, or: = $C(7, 4)$]

(b) 35 [There are $C(8, 4)$ 4-membered groups, and each such group determines a division of the 8 people into two groups of 4--
 A and \tilde{A} . However, A and \tilde{A} are two groups each of which determines the same division-- $\tilde{\tilde{A}} \neq A$, and dividing into A and \tilde{A} is the same as dividing into $\tilde{\tilde{A}}$ and A . So, the number of divisions is $\frac{1}{2} \cdot C(8, 4)$.]

(c) 70 [In contrast with part (b), we are now interested in ordered pairs of complementary groups of 4.]

(d) As in part (a), there are $C(2n - 1, n - 1)$ ways of dividing $2n - 1$ objects into a group of $n - 1$ objects and a group of n objects. As in part (b), there are $\frac{1}{2} \cdot C(2n, n)$ ways of dividing $2n$ objects into two groups of n objects. By Theorem 172,

$$\begin{aligned} C(2n - 1, n - 1) &= \frac{(2n - 1)!}{(n - 1)! n!} = \frac{(2n) \cdot (2n - 1)!}{(2n) \cdot (n - 1)! n!} \\ &= \frac{1}{2} \cdot \frac{(2n)!}{n! n!} = \frac{1}{2} \cdot C(2n, n). \end{aligned}$$

[Alternatively, by Theorem 173,

Corrections. On page 8-176, line 11b should read:
and 70 take math and French

line 8b should read:

(b) If less than 63 students---

and line 6b should read:

(c) If it is reported that 68 students---

Answers for Part A.

1. 25 [There is a one-to-one correspondence between round-trips and ordered pairs whose first components are up-routes and whose second components are down-routes. Since both U--the set of up-routes--and D--the set of down-routes are 5-membered sets, it follows by (*) on page 8-173 that $N(U \times D) = 5 \cdot 5$. Hence, by (C_1) , there are 25 round-trips.]
2. 30 [There is a one-to-one correspondence between the set of ways of dressing and the ordered triples with suits as first components, pairs of shoes as second components, and hats as third components. Defining, as is often done, ordered triples as ordered pairs whose first components are ordered pairs-- $(a, b, c) = ((a, b), c)$ --it follows, by (C_1) , that the number of ways of dressing is $N((A \times B) \times C)$, where A is a 5-membered set (of suits), B is a 3-membered set (of pairs of shoes), and C is a 2-membered set (of hats). By (*) page 8-173, $N(A \times B) = 5 \cdot 3$ and, so, $N((A \times B) \times C)$ is $(5 \cdot 3) \cdot 2$.

There is a more freewheeling way of obtaining the answer ' $(5 \cdot 3) \cdot 2$ ' by using principle (C_3) on page 8-177. But, it should do students no harm to see the solution of this problem brought back to first principles.

Of course, we assumed, in the solution, that the man does not wear the pants of his brown suit and the coat of his blue suit along with one brown shoe and one black shoe. But, with this more liberal interpretation of 'dressed' he can be dressed in 450 $[= 5 \cdot 5 \cdot 3 \cdot 3 \cdot 2]$ ways.]

3. (a) 252 $[= C(10, 5)]$ (b) 56 $[= C(8, 3)]$; 56 $[= C(8, 5)]$
(c) 100 $[= C(5, 2) \cdot C(5, 3)]$
(d) 126 $[= C(5, 1) \cdot C(5, 4) + C(5, 3) \cdot C(5, 2) + C(5, 5) \cdot C(5, 0)]$;
126 [The 5-person committees with an even number of men are the same as those with an odd number of women.];
126 [Since the total number (5) of women to choose from is the same as the total number of men, there is a one-to-one correspondence between the committees with an even number of women and those with an even number of men. Alternatively, the committees with an even number of women are just those without an odd number of women. So, of the former committees, there are $252 - 126$.]

1. 1980 The first of the two papers in this volume is by J. H. Conway and A. R. Meyer. It is a survey of the theory of the halting problem and the theory of the undecidable.

2. 1981 The second of the two papers in this volume is by J. H. Conway and A. R. Meyer. It is a survey of the theory of the halting problem and the theory of the undecidable.

3. 1982 The third of the two papers in this volume is by J. H. Conway and A. R. Meyer. It is a survey of the theory of the halting problem and the theory of the undecidable.

4. 1983 The fourth of the two papers in this volume is by J. H. Conway and A. R. Meyer. It is a survey of the theory of the halting problem and the theory of the undecidable.

5. 1984 The fifth of the two papers in this volume is by J. H. Conway and A. R. Meyer. It is a survey of the theory of the halting problem and the theory of the undecidable.

4. 4500 [He can send each son to any one of 5 schools and each daughter to any one of 6 schools. So, the number of ways is $5 \cdot 5 \cdot 5 \cdot 6 \cdot 6$.]
5. 7630 [$C(8, 5) \cdot C(5, 1) \cdot C(3, 1) + C(8, 4) \cdot C(5, 2) \cdot C(3, 1)$
 $+ C(8, 4) \cdot C(5, 1) \cdot C(3, 2) + C(8, 3) \cdot C(5, 3) \cdot C(3, 1)$
 $+ C(8, 3) \cdot C(5, 2) \cdot C(3, 2) + C(8, 3) \cdot C(5, 1) \cdot C(3, 3)]$
6. (a) 125 [$= 5 \cdot 5 \cdot 5$] (b) 60 [$= 5 \cdot 4 \cdot 3$] (c) 80 [$= 5 \cdot 4 \cdot 4$]
7. [The answers are the same as for Exercise 6 except that for part (a) there are 5 fewer arrangements because those in which all 3 pennants are the same color are, now, impossible. Alternatively, for part (a), there are $5 \cdot 1 \cdot 4$ arrangements in which the 2 top pennants are of the same color and $5 \cdot 4 \cdot 5$ in which the 2 top pennants are of different colors.]
8. 720 [$= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$]; 20160 [$= C(8, 6) \cdot 720$, or: $= 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$]
[See the discussion of Exercise 8 on page 8-179.]

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of all kinds, B is the set of round-trips in which the descent is by a different route than the ascent, and C is the set of round-trips in which the ascent and descent are by the same route, then $A = B \cup C$ and $B \cap C = \emptyset$. So, by (C_2) , $N(A) = N(B) + N(C)$. Since, by $(*)$, $N(A) = 5 \cdot 5$ and, by (C_1) , $N(C) = 5$, it follows that $N(B) = 5^2 - 5 = 20$.

*

Answers for Part B.

1. 2520 [= $C(10, 5) \cdot C(5, 3) \cdot C(2, 2)$, or: = $C(10, 3) \cdot C(7, 5) \cdot C(2, 2)$,
or: ...]
2. 15 [There are several ways of solving this exercise. We give three:
 - (1) Looking at any division of the 6 people into 3 pairs, one sees that it might be obtained in any of 3 ways--one might choose any one of the 3 pairs and then divide the remaining 4 people into 2 pairs. Thus, 3 times the number of ways in which 6 people can be divided into 3 pairs is $C(6, 2) \cdot \frac{1}{2} C(4, 2)$. So, the number of ways is $\frac{1}{3} \cdot 15 \cdot \frac{1}{2} \cdot 6$.
 - (2) Again looking at any division of the 6 people into 3 pairs, one sees that it might be obtained in one of 6 ways--one might choose any one of the 3 pairs, then any one of the remaining 2 pairs, and finally "choose" the pair that is left. So, the number of ways is $\frac{1}{6} \cdot C(6, 2) \cdot C(4, 2) \cdot C(2, 2)$.
 - (3) Begin by singling out a particular one of the 6 people. In any division, this person will be in one of the 3 pairs. This one pair is $\frac{1}{5}$ of 5 pairs and, when one of these 5 pairs has been chosen, all that remains is to divide the remaining 4 persons into 2 pairs. So, the number of ways is $5 \cdot \frac{1}{2} C(4, 2)$.]
3. 90 [= $C(6, 2) \cdot C(4, 2) \cdot C(2, 2)$]

line 12 on page 8-177, Explanation: You can pick the 4 person committee first, and then pick the 3-person committee from the remaining 11 people.

*

The counting principle (C_3) differs rather subtly from (*) on page 8-174. It can, as mentioned in the text, be derived from (C_1) and (C_2):

Let A be the set of ways in which the first event can occur and, for each $a \in A$, let B_a be the set of ways in which the second event can occur after the first event has occurred in way a . By hypothesis, $N(A) = m$ and $N(B_a) = n$. There is a one-to-one correspondence between the ways in which the two events can happen in succession and the ordered pairs (a, b) such that $a \in A$ and $b \in B_a$. So, by (C_1), the number of ways in which the two events can happen in succession is the number of members of $\{(a, b): a \in A \text{ and } b \in B_a\}$. This set of ordered pairs is the union of m disjoint sets, $\{(a, b): b \in B_a\}$, for $a \in A$. Since there is, for each $a \in A$, a one-to-one correspondence between the members of $\{(a, b): b \in B_a\}$ and the members of B_a , and since, for each $a \in A$, $N(B_a) = n$, it follows as in the derivation on page 8-174 of (*) that $N(\{(a, b): a \in A \text{ and } b \in B_a\})$ is mn .

Note that if, for each $a \in A$, $B_a = B$, then $\{(a, b): a \in A \text{ and } b \in B_a\} = A \times B$. In this case, the proof just given for (C_3) agrees with that given on page 8-174 for (*).

*

The problem of the mountaineer who must descend by a different route can be solved by using (C_3)--for each of 5 ways of ascending there are 4 ways of descending; so the number of round-trips is $5 \cdot 4$ --or by a combination of (*), (C_1), and (C_2). This second method, though more complicated than the first, is instructive. If A is the set of round-trips

and:

What is the number of combinations of 15 things, of which 8 are of one kind and 7 are of another kind, taken 5 at a time?

In general, two combinations [or permutations] are to be considered equivalent [and counted as one] if one can be obtained from the other by replacing one of the objects by another of the same kind; and two permutations are [also] to be considered equivalent if one can be obtained from the other by interchanging two objects which are of the same kind.

Coming, now, to the discussion in the text, the word 'needed' is to be considered as a set of 6 things, 3 of which are of one kind ['e's] and 2 of which are of another kind ['d's]. So, each permutation of the six letters is equivalent to 1 other [$1 = 2! - 1$] which can be obtained from it by interchanging the 'd's. And, each of a class of 2 such equivalent permutations is equivalent to 5 others [$5 = 3! - 1$] which can be obtained from it by permuting the 'e's. In sum, each of the $6!$ permutations of the 6 things is one of $2! 3!$ equivalent permutations, and the number of "different" permutations [strictly, the number of classes of equivalent permutations] is $6!/(2! 3!)$.

9. 60 [The 5 acts can be arranged in $5!$ ways. The 2 given acts will occur in the specified order in half of these $5!$ ways (and in the opposite order in the other half).]

10. (a) $720 [= P(6, 6)]$

(b) 60 [See the discussion following this exercise on page 8-182.]

*

In the discussion on page 8-182 and, to a greater extent in the optional material which begins on page 8-186, students encounter one of the prime sources of difficulties in interpreting combinatorial problems. Baldly put, the question is "When are two permutations [or combinations] the same?". The correct answer to this question, taken literally, is, of course, 'Never'. However, what is really involved is the question of deciding when two permutations or combinations are sufficiently similar that, for the purposes of the problem at hand, one should not distinguish between them. Let's illustrate this with two problems in combinations.

First, in how many ways can a committee of 5 people be chosen from a group of 15 people? As we know, the answer is ' $C(15, 5)$ ', and $C(15, 5) = 3003$.

Second, in how many ways can a handful of 5 nuts and bolts be chosen from a drawer containing 8 nuts (all of the same size, and none defective) and 7 bolts (all of the same size and none defective)? Here, again, we are choosing 5 things out of 15 and the answer might again be ' $C(15, 5)$ '. But, for most purposes, any collection consisting of, say, 3 of the nuts and 2 of the bolts would be as satisfactory as any other collection with the same composition. So, the expected answer is more likely to be '6' [5 bolts, or 1 nut and 4 bolts, or ... , or 5 nuts]. [But, if, as in Exercise 4 of Part A on page 8-175, one is "sampling" the contents of the drawer, the expected answer is, instead, ' $C(15, 5)$ '.] As illustrated by these two problems, when one sets up a problem, one usually knows what he is after, and knows what distinctions he wishes to take into account. The difficulty which a writer faces is that of expressing to a reader what these distinctions are. And, a reader has the problem of understanding the writer. The conventional solution of the writer's difficulty, as applied to the two problems above, is to state them as follows:

What is the number of combinations of 15
[different] things taken 5 at a time?

Answers for Part C.

1. 7980 $[= P(21, 3)]$
2. (a) $12!$ $[= P(12, 12)]$ (b) 103680 $[= 3!(4! 3! 5!)]$
3. $4!$ [Seat one person. Then, arrange the remaining 4 in, say, clockwise order around the table, starting from the seated person.]
4. (a) 240 $[= 5! \cdot 2! \text{ (See Exercise 2(b).)}]$ (b) 480 $[= P(6, 6) - 240]$
5. (a) 12144 $[= P(24, 3)]$ (b) 13824 $[= 24 \cdot 24 \cdot 24]$
6. 30240 $[= 2 \cdot 3 \cdot P(7, 7)]$
7. $(n - 2)! \cdot (n - p - 1) \cdot 2$ [Here are two ways to analyze this problem:
 - (1) The $n - 2$ "ungiven" people can be lined up in $(n - 2)!$ ways. Once this has been done, one of the given persons can be chosen in 2 ways, and this chosen person inserted in the line-up in any one of the $n - p - 1$ positions at which he will have at least p of the $n - 2$ people on his left. The remaining person then steps into the line-up at the left of the p th person to the left of the chosen one. The number of arrangements is $(n - 2)! \cdot 2 \cdot (n - p - 1)$.
 - (2) First, choose and arrange p of the $n - 2$ "ungiven" persons. This can be done in $P(n - 2, p)$ ways. Next, flank this group in either of 2 ways by the given people. There are now $n - p - 2$ people and one group of $p + 2$ people. These $n - p - 1$ things can be arranged in $(n - p - 1)!$ ways. The number of arrangements of the n people is $P(n - 2, p) \cdot 2 \cdot (n - p - 1)!$.
8. (a) $4!/2$ [Compare with Exercise 3. Since a ring can be turned over, each clockwise arrangement of keys on a ring [but, not of people at a table] is equivalent to a counterclockwise arrangement.]
 - (b) $(n - 1)!/2$ if $n > 2$; $(n - 1)!$ if $n \leq 2$ [It takes 3 points on a circle to distinguish a direction of rotation ('AC' vs. 'ABC').]
 - (c) 1 $[\neq (2 - 1)!/2]$

6. 3^9 [In succession, each of the 9 people exercises one of his 3 choices.]
7. 2520 [One way of bringing about such a distribution is to line up the 10 people in one of the possible $10!$ orders and send the first 5 into the living room, the next 3 into the dining room, and the last 2 into the kitchen. Arrangements in which the first 5 people are the same, the next 3 are the same, and the last 2 are the same, are, for this purpose, equivalent. So, the number of 'different' arrangements is $\frac{10!}{5! 3! 2!}$. Another way of bringing about such a distribution is to choose 5 to go into the living room and 3 for the dining room. So, the number of distributions is $C(10, 5) \cdot C(5, 3)$.]

8. The number of permutations of the kind described in Theorem 175 is

$$\begin{aligned}
 & C(p, p_1) \cdot C(p - p_1, p_2) \cdot \dots \cdot C(p - p_1 - \dots - p_{n-1}, p_n) \\
 & \quad \cdot (p - p_1 - \dots - p_n)! \\
 &= \frac{p!}{p_1! (p - p_1)!} \cdot \frac{(p - p_1)!}{p_2! (p - p_1 - p_2)!} \cdot \dots \cdot \frac{(p - p_1 - \dots - p_{n-1})!}{p_n! (p - p_1 - \dots - p_n)!} \\
 & \quad \cdot (p - p_1 - \dots - p_n)! \\
 &= \frac{p!}{p_1! p_2! \dots p_n!}
 \end{aligned}$$

*

line 36 on page 8-184: C_0 is the number of subsets of a 0-membered set. The only 0-membered set is \emptyset , and its only subset is \emptyset . So, $C_0 = 1$.

Answers for Part D.

1. (a) 554400 [or: $2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$] [$= \frac{11!}{3!2!3!}$]
 (b) 180 [$= \frac{6!}{2!2!}$] (c) 3360 [$= \frac{8!}{2!3!}$]
 (d) 9979200 [or: $2^5 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$] [$= \frac{12!}{2!2!3!2!} = \frac{12 \cdot 11! \cdot 3!}{2!2!3!2!3!} = 18 \cdot (a)$]
2. 10 [$= \frac{5!}{3!2!}$] [Notice that, by Theorem 175, if $n = 2$ and $p_1 + p_2 = p$, then the number of permutations in question is $C(p, p_1)$. This is to be expected, since such a problem amounts to finding the number of ways in which one can choose p_1 positions (out of p) for the p_1 things of the first kind.]
3. (a) 5040 [$= 7!$] (b) 210 [$= 7!/4!$] (c) 35 [$= 7!/(4!3!)$]
4. (a) 144 [To be seated alternately, the men must occupy the odd-numbered chairs (counting from either end) and the women must occupy the even numbered chairs. So (by courtesy) the women can be seated first in $3!$ ways, after which the men can be seated in $4!$ ways. So, by (C_3) , the answer is ' $3!4!$ '.]
 (b) 6 [$= 3!$] (c) 1152 [$= 2 \cdot 4!4!$]; 48 [$= 2 \cdot 4!$]
 (d) 144 [$= 3!4!$]; 6 [$= 3!$]
5. 1260 [$= \frac{9!}{4!3!2!}$] [In this exercise, the circularity of the table is irrelevant--the problem would be the same if the people were lined up in a row. If, in contrast, one asks for the number of ways the fruit could be distributed around the table before the chairs have been occupied (or place cards set), then the answer is: $\frac{8!}{4!3!2!}$]

How many "6-place" automobile license tags can be made if the first two places are letters and the last four are digits? [Ans: $26^2 \cdot 10^4$]

Bill has 3 pairs of shoes. In how many ways may he choose a right shoe and a left shoe? How many of these selections will not be mates? [Ans: 3^2 ; $3^2 - 3$]

In Zabbranchburg there are 6 elementary schools. In how many ways may 3 children attend these schools if no two are in the same school? [Ans: $P(6, 3)$]

In how many ways can a prowler enter a house by one window and leave by another if the house has 12 windows? [Ans: $12 \cdot 11$]

From a party of 6 boys and 4 girls, how many couples can be formed for dancing? [Ans: $6 \cdot 4$]

How many two-digit numerals can be formed from the digits '1', '2', '3', and '4'? [Ans: 4^2]

In how many ways may 6 boys be arranged in a line, if the same one always stands at the head of the line? [Ans: $5!$]

If the 8 questions on a test may be answered in any order, in how many orders is it possible to answer them? [Ans: $8!$]

Ten points are marked on a circle. How many inscribed triangles are there whose vertices belong to this set of ten points? [Ans: $C(10, 3)$]

An artist wishes to use 4 colors in drawing a poster. If he has 6 colors to choose from, how large a selection does he have? [Ans: $C(6, 4)$]

How many decimal numerals can be formed from the digits '2', '3', '4', '5', and '6', no digit being repeated? How many of these stand for even numbers?

$$[\text{Ans's: } 325 (= \sum_{p=1}^5 P(5, p)); 195 (= \sum_{p=1}^5 3 \cdot P(4, p-1))]$$

2. $2^{11} - 12$ [There are 2^{11} ways of choosing an odd-membered committee, but 12 such choices lead to 1-person committees.]
3. $(2^m - 1)(2^n - 1) + 1$ [If either of the factors of a cartesian product is \emptyset then so is the product. The number of nonempty cartesian products of the kind described is $(2^m - 1)(2^n - 1)$.]
4. $2^{m+n} - (2^m + 2^n - 1)$ [$A \cup B$ has 2^{m+n} subsets. 2^n of these are subsets of B--that is, contain no member of A. 2^m contain no member of B. 1, the empty set, is of both kinds. So, there are $2^m + 2^n - 1$ subsets each of which either contains no member of A or contains no member of B. The remaining subsets are those each of which contains at least one member of A and at least one member of B.]

*

Here are 15 counting problems, somewhat less challenging than those in the text. As a test of achievement, all students should be able to solve them.

How many four-letter words are there? [For the purposes of this problem, a four-letter word is what you get if you punch letter-keys on a typewriter four times.]

[Ans: 26^4]

How many integers are there between 2000 and 5000 whose decimal numerals can be formed from the digits '2', '4', '3', '9', and '7'?

[Ans: $3 \cdot 5 \cdot 5 \cdot 5$]

Find the number of combinations of 50 things taken 48 at a time.

[Ans: 1225]

In how many ways can the four aces of a deck of cards be arranged in a row?

[Ans: $4!(= 24)$]

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in the preceding discussion by references to binary numerals is that of making it obvious that different collections of the four weights have different weights.]

*

The proof of Theorem 177 which might be suggested [second sentence on page 8-186] by the proof on page 8-184 that $\forall_{j \geq 0} C_{j+1} = C_j \cdot 2$ goes as follows:

Suppose that S is a p -membered set, $e_0 \in S$, and S_0 is the $(p - 1)$ -membered subset of S which consists of the members of S other than e_0 . Then, each even-membered subset of S is either an even-membered subset of S_0 or is obtained by adjoining e_0 to an odd-membered subset of S_0 . Hence, S has just as many even-membered subsets as S_0 has subsets. Similarly, S has just as many odd-membered subsets as S_0 has subsets. Consequently, S has the same number of even-membered subsets as it has odd-membered subsets.

Note why the proof breaks down if S is \emptyset -in this case, there is no $e_0 \in S$.

The other proof of Theorem 177 which is referred to depends on the binomial theorem. See Exercise 2 of Part E on page 8-199.

*

Answers for Exercises.

1. 2^9 [The intersections of the 8 crossroads with either of the two roads divide the latter into 9 parts. There are 2^9 sets of these parts and there is a one-to-one correspondence between routes and sets of parts of one road, associating with each route the set consisting of those parts, of a given one of the roads, which the route traverses. (See Exercise 1(c) of the Miscellaneous Exercises beginning on page 7-91.)]

1. The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \int_0^x f(t) dt + \int_0^x g(t) dt \quad (1)$$

where $f(x)$ and $g(x)$ are continuous functions on the interval $[0, 1]$. It is shown that the function $f(x)$ is continuous on the interval $[0, 1]$ and that it satisfies the equation

$$f(x) = \int_0^x f(t) dt + \int_0^x g(t) dt \quad (2)$$

for all $x \in [0, 1]$. It is also shown that the function $f(x)$ is differentiable on the interval $[0, 1]$ and that its derivative is given by the equation

$$f'(x) = f(x) + g(x) \quad (3)$$

for all $x \in [0, 1]$. It is also shown that the function $f(x)$ is bounded on the interval $[0, 1]$ and that it attains its maximum and minimum values at the endpoints of the interval.

2. The second part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \int_0^x f(t) dt + \int_0^x g(t) dt \quad (4)$$

where $f(x)$ and $g(x)$ are continuous functions on the interval $[0, 1]$. It is shown that the function $f(x)$ is continuous on the interval $[0, 1]$ and that it satisfies the equation

$$f(x) = \int_0^x f(t) dt + \int_0^x g(t) dt \quad (5)$$

for all $x \in [0, 1]$. It is also shown that the function $f(x)$ is differentiable on the interval $[0, 1]$ and that its derivative is given by the equation

$$f'(x) = f(x) + g(x) \quad (6)$$

for all $x \in [0, 1]$. It is also shown that the function $f(x)$ is bounded on the interval $[0, 1]$ and that it attains its maximum and minimum values at the endpoints of the interval.

3. The third part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \int_0^x f(t) dt + \int_0^x g(t) dt \quad (7)$$

where $f(x)$ and $g(x)$ are continuous functions on the interval $[0, 1]$. It is shown that the function $f(x)$ is continuous on the interval $[0, 1]$ and that it satisfies the equation

$$f(x) = \int_0^x f(t) dt + \int_0^x g(t) dt \quad (8)$$

for all $x \in [0, 1]$.

Completion of proof of Theorem 176:

- (i) Since $C_0 = 1$, and, by the recursive definition of the exponential sequence with base 2, $2^0 = 1$, $C_0 = 2^0$.
- (ii) Suppose, for a $j \geq 0$, that $C_j = 2^j$. Since $C_{j+1} = C_j \cdot 2$ [line 3b on page 8-184], it follows that $C_{j+1} = 2^j \cdot 2 = 2^{j+1}$ [by the recursive definition of the exponential sequence with base 2]. Consequently, $\forall_{j \geq 0} [C_j = 2^j \Rightarrow C_{j+1} = 2^{j+1}]$.
- (iii) From (i) and (ii) it follows, by Theorem 114, that $\forall_{j \geq 0} C_j = 2^j$.

*

The "other proof" of Theorem 176 is based on the binomial theorem.
See Exercise 1 of Part E on page 8-199.

*

Line 10: If there are, for example, 30 students in your mathematics class then the number of situations referred to is 2^{30} .

*

Usually the word 'combination' is intended to suggest nonempty subsets.
So, one often finds the statement:

The total number of combinations of n things is $2^n - 1$.

*

The explanation of the weighing problem is very simple in terms of binary notation. For example, the binary numeral for the eighth-pound measure of the weight of, say, the one-pound weight, the half-pound weight, and the eighth-pound weight, together, is '1101'. So, the number of weights which can be measured is the number of at most 4-place binary numerals for positive integers. This is $2^4 - 1$. [The role played

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The generalizations of (5) and (6) referred to at the top of page 8-189 are:

(5*) The number of distributions of p things of the same kind in n boxes, at least q in each box, is $C(n - 1 + p - nq, n - 1)$.

and:

(6*) The number of combinations of p things of n kinds, at least q of each kind, is $C(n - 1 + p - nq, n - 1)$ [assuming that there are at least $p - (n - 1)q$ things of each kind to select from].

*

Explanation of answer for (c'): There are 3 ways in which the boy can dispose of his penny. Having done so, he has 3 ways in which to dispose of his nickel. So, by (C_3) , there are $3 \cdot 3$ ways to dispose of his penny and his nickel. Having done so, he has 3 ways to dispose of his dime. So, by (C_3) , he has $(3 \cdot 3) \cdot 3$ ways to dispose of his first three coins. Etc.

Explanation of answer for (d'): There are 3 ways in which one can decide on a kind of fruit for the first place in the line-up. Having done so, there are 3 ways to decide on a kind of fruit for the second place in the line-up. So, by (C_3) , there are $3 \cdot 3$ ways

*

Comparison of (3'), on page 8-190, with Theorem 176 should suggest that choosing a subset of a p -membered set can be likened to distributing p things in 2 boxes. The things which go into, say, the right-hand box constitute the subset one has chosen.

(b) $4! \cdot \frac{48!}{(12!)^4}$ [First, deal out the aces, then deal the remaining 48 cards.]

4. (a) $\frac{52!}{13!39!}$ [= C(52, 13)]

(b) $C(13, 5) \cdot C(39, 8)$

(c) $C(26, p) \cdot C(26, 13 - p)$

5. (a) 11550 [= $\frac{11!}{4!4!3!}$, by (5') on page 8-190]

(b) 5775 [= $\frac{11550}{2}$]

[The relation of part (b) to part (a) is the same as that of Exercise 8(b) of Part A on page 8-176 to Exercise 8(c).]

6. 1820 [= C(16, 4), by (4) on page 8-188]

7. 55 [= C(10 - 1 + 2, 10 - 1), by (3) or (4) on page 8-188]
[An alternative solution: There are 10^2 ordered pairs of numbers from 0 through 9. Each of 10 of these has the same number for both first and second component. So, there are $(10^2 - 10)/2$ unordered pairs with unequal components. So, counting doubles, there are $(10^2 - 10)/2 + 10$ dominoes.]

Answers to Exercises 8-12 are in the COMMENTARY for page 8-193.

line 3b on page 8-191: Why 5? Because the 4 blocks north which John has to walk begin or end on one of 5 east-west streets.

*

lines 4-6 on page 8-192: The answer to both questions is 'none'.

*

Answers for ☆Exercises.

1. 60 $\left[= \frac{6!}{2!3!1!} \right]$ [A throw of 6 dice may be thought of as a distribution of 6 things of different kinds into boxes--a 6-box, a 5-box, etc. So, the number in question is the number of distributions of 6 things of different kinds in 3 boxes, with 2 in the first box, 3 in the second, and 1 in the third.]

Alternatively, a throw of 6 dice may be thought of as an assignment of numbered tags to the dice. So, the number in question is the number of permutations of 6 things (one for each die) of 3 kinds--2 sixes, 3 fives, and 1 ace.]

2. (a) 6^{12} [the number of distributions of 12 different things (dice) in 6 boxes; the number of permutations of 12 things (numbered tags) of 6 kinds.]

- (b) $2^4 \cdot 3^5 \cdot 7 \cdot 11 \left[= \frac{12!}{(2!)^6} \right]$ [the number of distributions of 12 different things (dice) in 6 boxes, with 2 in each box; the number of permutations of 12 things (numbered tags) of 6 kinds, with 2 of each kind.]

3. (a) $\frac{52!}{(13!)^4}$ [the number of distributions of 52 different things (cards) in 4 boxes, with 13 in each box; the number of permutations of 52 things (players' name tags) of 4 kinds (players' names), with 13 of each kind.]

8. 455 [= $C(3+12, 3)$, by (4) on page 8-188]
9. 3003 [= $C(5+10, 5)$, by (3) on page 8-188]
10. 280; 280 [A sum of 3 positive integers is even if and only if either none or two of them are odd and is odd if and only if either none or two of them are even. Since, of 16 consecutive positive integers, 8 are even and 8 are odd, the same number of sets of 3 are even-summed as are odd-summed. So, the number of triples of either kind is $\frac{1}{2} \cdot C(16, 3)$.
Alternatively, the number of triples of either kind is $C(8, 3) + C(8, 2) \cdot C(8, 1)$.
Note that the restriction to positive integers is irrelevant.]
11. (a) 50388 [= $C(7+12, 7)$, by (4) on page 8-188]
(b) 1716 [= $C(7+6, 7)$, by (4) on page 8-188]
(c) 28 [= $C(8, 6)$, by (2) on page 3-187]
12. $C(n - p + 1, p)$ [Between each two successive members of a p -membered unfriendly set there is at least one positive integer. There may or may not be a positive integer which is less than each member of the set, and there may or may not be one which is greater than each member of the set and less than or equal to the given number n . An unfriendly set can be specified by telling how many integers are in each of these $(p - 1) + 2$ "boxes". So, there is a one-to-one correspondence between unfriendly p -membered sets each of whose members is at most n and distributions of $n - p$ things of the same kind in $p + 1$ boxes, a given $p - 1$ of which are nonempty. To determine the number of such distributions one proceeds as in deriving (5) on page 8-188 from (3). Put one of the $n - p$ things in each of the given $p - 1$ boxes and use (3) to determine the number of ways of distributing the remaining $n - 2p + 1$ things among the $p + 1$ boxes.]

and, since there are $C(10, 2)$ choices for p and q , $N_2 = 8! \cdot C(10, 2) = 10!/2!$. By similar arguments, $N_2 = 10!/3!, \dots$, and $N_{10} = 10!/10!$

Note that the probability of at least one letter being in its proper

envelope is $\sum_{p=1}^{10} \frac{(-1)^{p-1}}{p!}$. Since $\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} = \frac{1}{e}$ [where

$e = 2.71828\dots$], the probability in question is very nearly $1 - \frac{1}{e}$, or, approximately, $\frac{2}{3}$.]

1. The first part of the paper is devoted to the study of the

properties of the function $f(x)$ defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

where a_n are the coefficients of the power series. It is shown that

$f(x)$ is an entire function of order ρ and type σ if and only if

$$\limsup_{n \rightarrow \infty} \frac{\log a_n}{n} = \rho$$

and
$$\limsup_{n \rightarrow \infty} \frac{\log a_n}{n} = \sigma$$

where ρ and σ are the order and type of the function $f(x)$.

2. The second part of the paper is devoted to the study of the

properties of the function $f(x)$ defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

where a_n are the coefficients of the power series. It is shown that

$f(x)$ is an entire function of order ρ and type σ if and only if

$$\limsup_{n \rightarrow \infty} \frac{\log a_n}{n} = \rho$$

and
$$\limsup_{n \rightarrow \infty} \frac{\log a_n}{n} = \sigma$$

where ρ and σ are the order and type of the function $f(x)$.

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$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

where a_n are the coefficients of the power series. It is shown that

$f(x)$ is an entire function of order ρ and type σ if and only if

$$\limsup_{n \rightarrow \infty} \frac{\log a_n}{n} = \rho$$

and
$$\limsup_{n \rightarrow \infty} \frac{\log a_n}{n} = \sigma$$

line 6: For i odd, $(-1)^{i-1} = 1$; and, for i even, $(-1)^{i-1} = -1$.

*

Answers for ☆Exercises.

1. 100 [Let the sets of families in the four categories for which data are given be A_1, A_2, A_3 , and A_4 , respectively. Then, $A_2 \cap A_3$ is the set of families each of which has an annual income of at least \$3000 but less than \$4000. Except for $A_2 \cap A_3$, each intersection of two or more of the 4 sets is empty. Hence, in the notation of (C_4) , $N_2 = N(A_2 \cap A_3)$ and $N_3 = N_4 = 0$. From the given data, $N_1 = 900 + 2000 + 900 + 50 = 3850$. So,

$$3050 = 3850 - N(A_2 \cap A_3),$$

and $N(A_2 \cap A_3) = 800$. Since the number of families with incomes of at least \$3000 but less than \$5000 is 900, it follows that the number with incomes of at least \$4000 but less than \$5000 is $900 - 800$.

Other exercises of this kind can be found in the article "Problems involving overlapping finite sets" by Brother U. Alfred, F.S.C. in The Mathematics Teacher for November, 1960. However, note the errors which are pointed out on page 335 of the May 1961 issue.]

2. $10! \cdot \sum_{p=1}^{10} \frac{(-1)^{p-1}}{p!}$ [For $1 \leq p \leq 10$, let A_p be the set of those ways putting the letters in envelopes so that the p th letter is in the proper envelope. Evidently, $N(A_p) = 9!$. Since there are 10 choices for p , $N_1 = 9! \cdot 10 = 10!$ Also, for $p \neq q$, $N(A_p \cap A_q) = 8!$

$$\begin{aligned}
 \text{line 6: } (a + b)^3(a + b) &= (a^3 + 3a^2b + 3ab^2 + b^3)(a + b) \\
 &= a^4 + 3a^3b + 3a^2b^2 + ab^3 + a^3b + 3a^2b^2 + 3ab^3 + b^4 \\
 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
 \end{aligned}$$

$$\text{line 8: } (a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

lines 13-18: Have students tell which of the dpma and the ldpma justifies each step:

(1) - (2) [dpma], (2) - (3) [ldpma], (3) - (4) [dpma],
 (4) - (5) [dpma], (5) - (6) [ldpma]

line 23: 16

$\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 x}{dt^2} \right) = \frac{1}{2} \frac{d^3 x}{dt^3}$
 $\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 x}{dt^2} \right) = \frac{1}{2} \frac{d^3 x}{dt^3}$
 $\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 x}{dt^2} \right) = \frac{1}{2} \frac{d^3 x}{dt^3}$

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$\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \frac{d^2 x}{dt^2} \right) = \frac{1}{2} \frac{d^3 x}{dt^3}$

*

Answers for Part B.

1. $-14784x^5$ $[= C(11, 5)1^{11-5}(-2x)^5 = 462 \cdot 1 \cdot -32x^5]$
2. $262440x^7y^3$ $[= C(10, 3)(3x)^{10-3}y^3 = 120 \cdot 3^7x^7y^3]$
3. $36a^4b^7c^7$ $[= C(9, 7)(a^2)^{9-7}(bc)^7]$
4. $-583649763840u^4v^9$ $[= C(13, 9)(3u)^{13-9}(-6v)^9 = -715 \cdot 3^46^9u^4v^9]$
5. $\frac{5}{54}x^3y^3$ $[= C(6, 3)(\frac{x}{2})^{6-3}(\frac{y}{3})^3 = 20 \cdot \frac{x^3}{8} \cdot \frac{y^3}{27}]$
6. 1120 $[= C(8, 4)x^{8-4}(\frac{2}{x})^4 = 70 \cdot x^4 \cdot \frac{16}{x^4}]$
7. $10^{16}x^{16}$ 8. $97 \cdot 2^{96}pq^{96}$

Answers for Exercises 9 and 10 of Part B are in the COMMENTARY for page 8-199.

1867
The first of the year was a very dry one, and the crops were much injured. The weather was very hot, and the ground was very dry. The crops were much injured, and the people were very poor. The weather was very hot, and the ground was very dry. The crops were much injured, and the people were very poor.

The second of the year was a very wet one, and the crops were much injured. The weather was very cold, and the ground was very wet. The crops were much injured, and the people were very poor. The weather was very cold, and the ground was very wet. The crops were much injured, and the people were very poor.

The third of the year was a very dry one, and the crops were much injured. The weather was very hot, and the ground was very dry. The crops were much injured, and the people were very poor. The weather was very hot, and the ground was very dry. The crops were much injured, and the people were very poor.

The fourth of the year was a very wet one, and the crops were much injured. The weather was very cold, and the ground was very wet. The crops were much injured, and the people were very poor. The weather was very cold, and the ground was very wet. The crops were much injured, and the people were very poor.

The fifth of the year was a very dry one, and the crops were much injured. The weather was very hot, and the ground was very dry. The crops were much injured, and the people were very poor. The weather was very hot, and the ground was very dry. The crops were much injured, and the people were very poor.

The sixth of the year was a very wet one, and the crops were much injured. The weather was very cold, and the ground was very wet. The crops were much injured, and the people were very poor. The weather was very cold, and the ground was very wet. The crops were much injured, and the people were very poor.

The seventh of the year was a very dry one, and the crops were much injured. The weather was very hot, and the ground was very dry. The crops were much injured, and the people were very poor. The weather was very hot, and the ground was very dry. The crops were much injured, and the people were very poor.

The eighth of the year was a very wet one, and the crops were much injured. The weather was very cold, and the ground was very wet. The crops were much injured, and the people were very poor. The weather was very cold, and the ground was very wet. The crops were much injured, and the people were very poor.

line 4: Note that 'terms' is used here to refer to certain expressions. This usage differs from that on page 8 - 2ff. where the terms of a sequence are the values of the sequence [i. e. function]. Words like 'term', 'exponent', 'coefficient' etc. are customarily used ambiguously and we shall, at least to some extent, follow the custom.

*

An example of the short cut:

$$(x + -y)^6 = 1x^6 + 6x^5(-y) + 15x^4(-y)^2 + 20x^3(-y)^3 + \dots$$

*

Answers for Part A [which begins on page 8-197].

1. $a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$

2. $x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$

3. $16a^4 + 32a^3b + 24a^2b^2 + 8ab^3 + b^4$

4. $243x^5 - 810x^4y + 1080x^3y^2 - 720x^2y^3 + 240xy^4 - 32y^5$

5. $16807z^5 - 12005z^4 + 3430z^3 - 490z^2 + 35z - 1$

6. $\frac{x^5}{32} + \frac{x^4y}{16} + \frac{x^3y^2}{20} + \frac{x^2y^3}{50} + \frac{xy^4}{250} + \frac{y^5}{3125}$

7. $252 - 144\sqrt{3}$

8. $t^7 + 7t^5 + 21t^3 + 35t + 35t^{-1} + 21t^{-3} + 7t^{-5} + t^{-7}$

9. $485 + 258\sqrt{6}$

1. The first part of the paper is devoted to the study of the

2. The second part of the paper is devoted to the study of the

3. The third part of the paper is devoted to the study of the

4. The fourth part of the paper is devoted to the study of the

5. The fifth part of the paper is devoted to the study of the

6. The sixth part of the paper is devoted to the study of the

7. The seventh part of the paper is devoted to the study of the

8. The eighth part of the paper is devoted to the study of the

9. The ninth part of the paper is devoted to the study of the

10. The tenth part of the paper is devoted to the study of the

Hence, for $|x| < 1$, if $\sum_{k=0}^{\infty} \binom{j}{k} x^k = (1+x)^j$

then $\sum_{k=0}^{\infty} \binom{j-1}{k} x^k = (1+x)^{j-1}$.

This is an instructive bit of algebra, but is not all that is necessary for part (ii) of the proof. What else is required is a preliminary paragraph in which it is shown, for $|x| < 1$, that the sum sequences whose n th

terms are given by $\sum_{k=0}^n \binom{j}{k} x^k$, and $\sum_{k=0}^n \binom{j-1}{k} x^k$, do have limits. [That

this is necessary is witnessed to by the fact that, for $|x| \geq 1$, they don't.] The "formal part" of the proof has been given, above, in case any of your brighter students should discover it. If this happens, he should, of course, be congratulated. But, you should point out to him that, when dealing with "infinite sums", one must begin by making sure that they "exist".

The preceding discussion may raise questions about nonintegral exponents. Once the concepts have been defined, it is possible to prove [again, for $|x| < 1$] that

$$\forall_z (1+x)^z = \sum_{k=0}^{\infty} \binom{z}{k} x^k.$$

However, more than induction is needed for this.

*

line 1 on page 8-198: Values of the function C are often, as here, called 'binomial coefficients'.

$$\begin{aligned}
\text{Then } (1+x) \sum_{k=0}^{\infty} \binom{j-1}{k} x^k &= \sum_{k=0}^{\infty} \binom{j-1}{k} x^k + \sum_{k=0}^{\infty} \binom{j-1}{k} x^{k+1} \\
&= \sum_{k=0}^{\infty} \binom{j-1}{k} x^k + \sum_{k=1}^{\infty} \binom{j-1}{k-1} x^k \\
&= \binom{j-1}{0} x^0 + \sum_{k=1}^{\infty} \binom{j-1}{k} x^k + \sum_{k=1}^{\infty} \binom{j-1}{k-1} x^k \\
&= \binom{j-1}{0} x^0 + \sum_{k=1}^{\infty} \left[\binom{j-1}{k} + \binom{j-1}{k-1} \right] x^k \\
&= \binom{j-1}{0} x^0 + \sum_{k=1}^{\infty} \binom{j}{k} x^k \\
&= \binom{j}{0} x^0 + \sum_{k=1}^{\infty} \binom{j}{k} x^k \\
&= \sum_{k=0}^{\infty} \binom{j}{k} x^k \\
&= (1+x)^j.
\end{aligned}$$

So [since $1+x \neq 0$]

$$\sum_{k=0}^{\infty} \binom{j-1}{k} x^k = (1+x)^j / (1+x) = (1+x)^{j-1}.$$

The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \frac{1}{x} \int_0^x f(t) dt$$

It is shown that the function $f(x)$ is continuous and differentiable for all $x \neq 0$. The derivative of $f(x)$ is given by the formula

$$f'(x) = -\frac{1}{x^2} \int_0^x f(t) dt + \frac{f(x)}{x}$$

It is also shown that the function $f(x)$ satisfies the differential equation

$$x^2 f''(x) + x f'(x) - f(x) = 0$$

The general solution of this equation is given by the formula

$$f(x) = \frac{1}{x} \int_0^x f(t) dt + C_1 x + C_2 x^{-1}$$

where C_1 and C_2 are arbitrary constants. The function $f(x)$ is determined by the initial conditions

$$f(1) = 1, \quad f'(1) = 0$$

is no hindrance to extending the definition of C by relaxing all restrictions on its first argument. When this is done, it is customary to adopt a different notation for the new function-- $\binom{z}{k}$ in place of ' $C(z, k)$ '--and define it recursively by:

$$\begin{cases} \forall_z \binom{z}{0} = 1 \\ \forall_z \forall_{k \geq 0} \binom{z}{k+1} = \binom{z}{k} \cdot \frac{z-k}{k+1} \end{cases}$$

The proof [on page 8-170] of Theorem 171 goes through as before, yielding the theorem:

$$\forall_z \forall_{k \geq 0} \binom{z}{k} = \frac{\prod_{i=0}^{k-1} (z-i)}{k!},$$

and the proof in the answer for Part $\star I$ on page 8-172 also goes through, and yields:

$$\forall_z \forall_n \binom{z}{n} = \binom{z-1}{n} + \binom{z-1}{n-1}$$

Using this extension of Theorem 173 one can construct part (ii) of a "backward" inductive proof [Theorem 116] to show, for $|x| < 1$, that

$$(**) \quad \forall_{j \leq -1} (1+x)^j = \sum_{k=0}^{\infty} \binom{j}{k} x^k.$$

[Part (i) of the proof comes, of course, from (*).] Formally, the proof of (**) is quite straightforward. In brief [part (ii)]:

$$\text{Suppose, for } |x| < 1, \text{ that } \sum_{k=0}^{\infty} \binom{j}{k} x^k = (1+x)^j.$$

contains ' $\sum_{k=0}^j$ ', [which, with '-1' for 'j', becomes ' $\sum_{k=0}^{-1}$ '] instead of

' $\sum_{k=0}^{\infty}$ ', may be "explained" by recalling Exercise ☆2 of Part D on page

8-169. When j is a nonnegative integer then $C(j, k) = 0$ for $k \geq j + 1$. So

Theorem 178 might, itself, be written with ' $\sum_{k=0}^{\infty}$ ' instead of ' $\sum_{k=0}^j$ '.

The need for the restriction ' $|x| < 1$ ' has been brought out on page 8-143.

Briefly, it is due to the fact that, while any finite number of terms of a sequence have a sum, infinitely many terms of a sequence may fail to have a sum--"an infinite sum may fail to exist". [Recall (page 8-138) that the second word 'sum' in the preceding sentence has a different meaning than does the first.] So, although no restriction is needed on 'x' and 'y' in Theorem 178, a restriction happens to be needed in Theorem 168b and its equivalent, (*).

[One can use (*) to derive a theorem a little more like an instance of Theorem 178. In fact, for $x \neq 0$ and $|x| < |y|$,

$$(x+y)^{-1} = x^{-1} \left(1 + \frac{y}{x}\right)^{-1} = \underset{(*)}{x^{-1}} \sum_{k=0}^{\infty} C(-1, k) \left(\frac{y}{x}\right)^k = \sum_{k=0}^{\infty} C(-1, k) x^{-1-k} y^k$$

Besides "algebra", including laws of exponents, and (*), one needs an analogue, for infinite sums, of Theorem 133.]

As students should be quick to guess, similar results hold for binomial exponentials with any negative integral exponents. To begin with, there

and conclude, in line 2b, that $\sum_{k=0}^{\infty} a_{k+1} = \frac{a_1}{1-r}.$]

Now, if, in Theorem 171, we ignore the restriction ' $j \geq 0$ ', we can obtain:

$$\begin{aligned}
 C(-1, k) &= \frac{\prod_{i=0}^{k-1} (-1-i)}{k!} \\
 &= \frac{\prod_{i=0}^{k-1} (-1) \cdot \prod_{i=0}^{k-1} (1+i)}{k!} \\
 &= \frac{\prod_{i=1}^k (-1) \cdot \prod_{i=1}^k i}{k!} \\
 &= \frac{(-1)^k k!}{k!} = (-1)^k
 \end{aligned}
 \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \begin{array}{l} \text{Theorems 28 and 145} \\ \\ \text{Theorem 148} \end{array}$$

[Of course, the numbers $C(-1, k)$ have nothing to do with counting subsets.]

So, if we adopt Theorem 171, with ' $\forall_{j \geq 0}$ ' replaced by ' \forall_j ', as a definition, it turns out that, for $|x| < 1$,

$$(*) \quad (1+x)^{-1} = \sum_{k=0}^{\infty} C(-1, k)x^k.$$

This last compares favorably with the binomial theorem ['1' for 'x', 'x' for 'y', and '-1' for 'j' in Theorem 178]. The fact that Theorem 178

1. The first of these is the fact that the
 2. second of these is the fact that the
 3. third of these is the fact that the
 4. fourth of these is the fact that the
 5. fifth of these is the fact that the

6. The first of these is the fact that the
 7. second of these is the fact that the
 8. third of these is the fact that the
 9. fourth of these is the fact that the
 10. fifth of these is the fact that the

11. The first of these is the fact that the

12. The first of these is the fact that the

13. The first of these is the fact that the

14. The first of these is the fact that the

15. The first of these is the fact that the

16. The first of these is the fact that the

17. The first of these is the fact that the

18. The first of these is the fact that the
 19. The first of these is the fact that the
 20. The first of these is the fact that the

21. The first of these is the fact that the

line 6 on page 8-197: $(a + b)^n = \sum_{k=0}^n C(n, k)a^{n-k}b^k$

line 7: Called 'the binomial theorem' because an expression whose principle operator is a '+' is a binomial. For example, '-6(a + b) + -2c' is a binomial.



*

Students may wonder whether there is a binomial theorem for negative integral exponents. There is, and they are already acquainted with an instance of it. By Exercise 2 of Part E on page 8-145, for $|x| < 1$, $|-x| < 1$, and, by Theorem 168b, for $|-x| < 1$,

$$\begin{aligned} (1 + x)^{-1} &= (1 - (-x))^{-1} \\ &= \sum_{p=0}^{\infty} (-x)^{p-1} \\ &= \sum_{k=0}^{\infty} (-x)^k \\ &= \sum_{k=0}^{\infty} (-1)^k x^k. \end{aligned} \quad \left. \begin{array}{l} \left. \begin{array}{l} \text{Theorem 168b} \\ \text{Theorem 137} \end{array} \right\} \\ \text{Theorems 28 and 155} \end{array} \right\}$$

[The reference to Theorem 137 is not quite appropriate--the theorem needed here is one like Theorem 137, but with ' $\forall_{k \geq j-1}$ ' deleted and 'k' and ' $k+j_1$ ' replaced by ' ∞ '. Alternatively, one can apply Theorem 137

to modify line 4b on page 8-146, changing ' $\sum_{p=1}^n a_1 r^{p-1}$ ' to ' $\sum_{k=0}^{n-1} a_1 r^k$ ',

1. The first of these is the fact that the number of
of the population is not constant.

2. The second is the fact that the number of
of the population is not constant.

3. The third is the fact that the number of
of the population is not constant.

4. The fourth is the fact that the number of
of the population is not constant.

5. The fifth is the fact that the number of
of the population is not constant.

The third step on page 8-200 follows from the second by Theorem 137 and some algebra $[j - (k - 1) = (j + 1) - k]$.

The fourth step [two lines of print] follows from the third by an application of Theorem 136 [to the first sum] and an application of the recursive definition of Σ -notation [to the second sum].

The fifth step follows from the fourth because $C(j, 0) = 1 = C(j, j)$ and by an application of [the apa and] Theorem 134 [and the dpma].

The sixth step follows from the fifth by Theorem 173 because $C(j+1, 0) = 1 = C(j+1, j+1)$.

The last step follows by Theorem 136 and the recursive definition of Σ -notation.

$$2. \quad 0 = 0^n = (1 + -1)^n = \sum_{k=0}^n C(n, k) 1^{n-k} (-1)^k = \sum_{k=0}^n C(n, k) (-1)^k$$

= the number of even-membered subsets of an n-membered set minus the number of its odd-numbered subsets.

$$3. \quad (1 + a)^0 = 1 = 1 + 0a$$

$$(1 + a)^n = \sum_{k=0}^n C(n, k) a^k = C(n, 0) a^0 + C(n, 1) a^1 + \sum_{k=2}^n C(n, k) a^k.$$

Since $C(n, k) a^k \geq 0$ for $a \geq 0$ and $k \geq 2$, it follows, using Theorems 143 and 133, that $\sum_{k=2}^n C(n, k) a^k \geq 0$. Hence, it follows, for $a \geq 0$,

$$\text{that } (1 + a)^n \geq C(n, 0) a^0 + C(n, 1) a^1 = 1 + na.$$

$$\text{Consequently, } \forall_{x \geq 0} \forall_{k \geq 0} (1 + x)^k \geq 1 + kx.$$

[Notice that the case $k = 0$ had to be treated separately.]

*

Answer for Part F.

Part (i) is justified by three recursive definitions--that for exponential sequences [page 8-101], that for C [page 8-168], and that for Σ -notation [page 8-36].

The first displayed line in part (ii) is, of course, the inductive hypothesis, and the last line on page 8-199 follows from this and the recursive definition of exponential sequences [and the cpm].

The first step on page 8-200 follows by the dpma and the next follows by Theorem 133, the recursive definition of exponential sequences, and some algebra [cpm, apm, etc.].

1. The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \int_0^x f(t) dt$. It is shown that $f(x)$ is a constant function, and its value is determined by the initial condition $f(0) = 1$.

2. In the second part, we consider the function $g(x)$ defined by the equation $g(x) = \int_0^x g(t) dt$. It is shown that $g(x)$ is a constant function, and its value is determined by the initial condition $g(0) = 1$.

3. The third part of the paper is devoted to the study of the properties of the function $h(x)$ defined by the equation $h(x) = \int_0^x h(t) dt$. It is shown that $h(x)$ is a constant function, and its value is determined by the initial condition $h(0) = 1$.

4. In the fourth part, we consider the function $k(x)$ defined by the equation $k(x) = \int_0^x k(t) dt$. It is shown that $k(x)$ is a constant function, and its value is determined by the initial condition $k(0) = 1$.

5. The fifth part of the paper is devoted to the study of the properties of the function $l(x)$ defined by the equation $l(x) = \int_0^x l(t) dt$. It is shown that $l(x)$ is a constant function, and its value is determined by the initial condition $l(0) = 1$.

6. In the sixth part, we consider the function $m(x)$ defined by the equation $m(x) = \int_0^x m(t) dt$. It is shown that $m(x)$ is a constant function, and its value is determined by the initial condition $m(0) = 1$.

7. The seventh part of the paper is devoted to the study of the properties of the function $n(x)$ defined by the equation $n(x) = \int_0^x n(t) dt$. It is shown that $n(x)$ is a constant function, and its value is determined by the initial condition $n(0) = 1$.

8. In the eighth part, we consider the function $o(x)$ defined by the equation $o(x) = \int_0^x o(t) dt$. It is shown that $o(x)$ is a constant function, and its value is determined by the initial condition $o(0) = 1$.

9. The ninth part of the paper is devoted to the study of the properties of the function $p(x)$ defined by the equation $p(x) = \int_0^x p(t) dt$. It is shown that $p(x)$ is a constant function, and its value is determined by the initial condition $p(0) = 1$.

10. In the tenth part, we consider the function $q(x)$ defined by the equation $q(x) = \int_0^x q(t) dt$. It is shown that $q(x)$ is a constant function, and its value is determined by the initial condition $q(0) = 1$.

to find the expansion of, say, $(a + b + c)^{10}$. As in the case of $(a + b)^4$ [see page 8-196], $(a + b + c)^{10}$ is, by the dpma and the ldpma, equivalent to an indicated sum of 3^{10} 10-letter product-expressions. [3^{10} because each 10-letter product corresponds with a distribution of 10 things (factors) in 3 boxes (an 'a'-box, a 'b'-box, and a 'c'-box. By (3') on page 8-190, the number of such distributions is 3^{10} .] Now, if j_1, j_2 , and j_3 are nonnegative integers such that $j_1 + j_2 + j_3 = 10$ then the number of these 10-letter product-expressions which contain j_1 'a's, j_2 'b's, and j_3 'c's is the number of distributions of 10 things in 3 boxes, with, for $1 \leq i \leq 3$, j_i things in the box--that is, by (5') on page 8-190,

$\frac{10!}{i_1! i_2! i_3!}$. So, as in the case of the binomial expansion,

$$(a + b + c)^{10} = \sum \frac{10!}{i_1! i_2! i_3!} a^{i_1} b^{i_2} c^{i_3},$$

where the summation extends to all terms for which i_1, i_2 , and i_3 are nonnegative integers whose sum is 10.

In general,

$$\forall_{j \geq 0} \left(\sum_{p=1}^m a_p \right)^j = \sum \frac{j!}{\prod_{p=1}^m i_p!} \prod_{p=1}^m a_p^{i_p},$$

where the summation extends to all terms for which, for $1 \leq p \leq m$, i_p

is a nonnegative integer, and $\sum_{p=1}^m i_p = j$.

*

Answers for Part E.

$$1. C_j = \sum_{k=0}^j C(j, k) = \sum_{k=0}^j C(j, k) 1^{j-k} 1^k = (1 + 1)^j = 2^j$$

4. 0.668

| | | | | |
|---|------------------------------------|---|----------------------------|-----------------------|
| [| 1^{20} | = | 1 | 1 |
| | $-20 \cdot 2 \cdot 10^{-2}$ | = | $-40 \cdot 10^{-2}$ | 0.6 |
| | $190 \cdot 2^2 \cdot 10^{-4}$ | = | $760 \cdot 10^{-4}$ | 0.676 |
| | $-1140 \cdot 2^3 \cdot 10^{-6}$ | = | $-9120 \cdot 10^{-6}$ | 0.66688 |
| | $4845 \cdot 2^4 \cdot 10^{-8}$ | = | $77520 \cdot 10^{-8}$ | 0.6676552 |
| | $-15504 \cdot 2^5 \cdot 10^{-10}$ | = | $-496128 \cdot 10^{-10}$ | 0.6676055872 |
| | $38760 \cdot 2^6 \cdot 10^{-12}$ | = | $2480640 \cdot 10^{-12}$ | 0.66760806784 |
| | $-77520 \cdot 2^7 \cdot 10^{-14}$ | = | $-9922560 \cdot 10^{-14}$ | 0.6676079686144 |
| | $125970 \cdot 2^8 \cdot 10^{-16}$ | = | $32248320 \cdot 10^{-16}$ | 0.667607971839232 |
| | $-167960 \cdot 2^9 \cdot 10^{-18}$ | = | $-85995520 \cdot 10^{-18}$ | 0.66760797175323648] |

✱

Answers for Part D.

$$\begin{aligned}
 1. \quad (x + y + z)^2 &= (x + y)^2 + 2(x + y)z + z^2 \\
 &= x^2 + 2xy + y^2 + 2xz + 2yz + z^2 \\
 &= x^2 + y^2 + z^2 + 2(xy + yz + zx)
 \end{aligned}$$

$$\begin{aligned}
 2. \quad (1 + a + b)^3 &= (1 + a)^3 + 3(1 + a)^2b + 3(1 + a)b^2 + b^3 \\
 &= 1 + 3a + 3a^2 + a^3 + 3b + 6ab + 3a^2b + 3b^2 + 3ab^2 + b^3 \\
 &= 1 + a^3 + b^3 + 3(a + a^2b + b^2) + 3(a^2 + ab^2 + b) + 6ab
 \end{aligned}$$

$$3. \quad 1 + 4x + 10x^2 + 16x^3 + 19x^4 + 16x^5 + 10x^6 + 4x^7 + x^8$$

✱

The binomial theorem is a special case of a "multinomial theorem". Note that the coefficients of the term containing ' $a^j - b^k$ ' in the expansion of ' $(a + b)^j$ ' is the number of ways of distributing j things (the j factors) into 2 boxes (an ' a '-box and a ' b '-box) with $j - k$ things in the first box and k in the second. This suggests using (5') on page 8-190

$$2. \quad 0.886 [(1 - 10^{-2})^{12}]$$

$$= 1^{12} - 12 \cdot 10^{-2} + 66 \cdot 10^{-4} - 220 \cdot 10^{-6} + 495 \cdot 10^{-8} - 792 \cdot 10^{-10} + \dots$$

$$= 1 - 0.12 + 0.0066 - 0.00022 + 0.00000495 - 0.0000000792 + \dots$$

$$0.99^{12} \doteq 1 \quad [1 \text{ term}]$$

$$0.99^{12} \doteq 0.88 \quad [2 \text{ terms}]$$

$$0.99^{12} \doteq 0.8866 \quad [3 \text{ terms}]$$

$$0.99^{12} \doteq 0.88638 \quad [4 \text{ terms}]$$

$$0.99^{12} \doteq 0.88638495 \quad [5 \text{ terms}]$$

$$0.99^{12} \doteq 0.8863848708 \quad [6 \text{ terms}]$$

Students may be interested to notice that in a case like this, when the successive terms are alternately positive and negative and decrease in absolute value, the error at each stage is less than the absolute value of the next term. For example,

$$-0.12 < 0.99^{12} - 1 < 0$$

$$0 < 0.99^{12} - 0.88 < .0066$$

$$-0.00022 < 0.99^{12} - 0.8866 < 0$$

$$\text{etc.} \quad]$$

$$3. \quad 1.344 [(1 + 3 \cdot 10^{-2})^{10}]$$

$$= 1^{10} + 10 \cdot 3 \cdot 10^{-2} + 45 \cdot 3^2 \cdot 10^{-4} + 120 \cdot 3^3 \cdot 10^{-6} + 210 \cdot 3^4 \cdot 10^{-8}$$

$$+ 252 \cdot 3^5 \cdot 10^{-10} + 210 \cdot 3^6 \cdot 10^{-12} + 120 \cdot 3^7 \cdot 10^{-14} + \dots$$

$$= 1 + 0.3 + 0.0385 + 0.00324 + 0.0001701 + 0.0000061236$$

$$+ 0.00000015309 + 0.0000000026244 + \dots$$

$$(1.03)^{10} \doteq 1 \quad [1 \text{ term}]$$

$$(1.03)^{10} \doteq 1.3 \quad [2 \text{ terms}]$$

$$(1.03)^{10} \doteq 1.3405 \quad [3 \text{ terms}]$$

$$(1.03)^{10} \doteq 1.34174 \quad [4 \text{ terms}]$$

$$(1.03)^{10} \doteq 1.3419101 \quad [5 \text{ terms}]$$

$$(1.03)^{10} \doteq 1.3419162236 \quad [6 \text{ terms}]$$

$$(1.03)^{10} \doteq 1.34191637669 \quad [7 \text{ terms}]$$

$$(1.03)^{10} \doteq 1.3419163793144 \quad [8 \text{ terms}] \quad]$$

$$9. -252x^5 \left[= C(10, 5) \left(\frac{x^2}{2} \right)^{10-5} \left(-\frac{2}{x} \right)^5 \right]$$

$$10. 280y^6 \left[= C(7, 3) 1^{7-3} (2y^2)^3 = 35 \cdot 8y^6 \right]$$

*

In part C students are asked to compute approximations which are correct to the nearest 0.001. This amounts [see TC[3-116] of Unit 3] to obtaining 3-decimal place approximations which are in error by less than 0.0005. Students are expected, in doing these exercises, to decide by guess when they have added up enough terms to obtain a sufficiently good approximation. Actually, for small values of 'x', the values of the successive terms in the binomial expansion of '(1 + x)ⁿ' decrease so rapidly that their guesses are very likely to be correct.

*

Answers for Part C.

1. 1.707 [Here is more computing than one actually needs.

$$\begin{aligned} 1.02^{27} &= 1^{27} + 27(0.02) + 351(0.02)^2 + 2925(0.02)^3 \\ &\quad + 17550(0.02)^4 + 80730(0.02)^5 + 296010(0.02)^6 \\ &\quad + 888030(0.02)^7 + 2220075(0.02)^8 + \dots \\ &= 1 + 0.54 + 0.1404 + 0.0234 + 0.002808 \\ &\quad + 0.000258336 + 0.00001894464 \\ &\quad + 0.0000011366784 + 0.00000005683392 + \dots \end{aligned}$$

| | |
|-------------------------------------|-------------------|
| $1.02^{27} \doteq 1$ | [1 term] |
| $1.02^{27} \doteq 1.54$ | [2 terms] |
| $1.02^{27} \doteq 1.6804$ | [3 terms] |
| $1.02^{27} \doteq 1.7038$ | [4 terms] |
| $1.02^{27} \doteq 1.706608$ | [5 terms] |
| $1.02^{27} \doteq 1.706866336$ | [6 terms] |
| $1.02^{27} \doteq 1.70688528064$ | [7 terms] |
| $1.02^{27} \doteq 1.7068864173184$ | [8 terms] |
| $1.02^{27} \doteq 1.70688647415232$ | [9 terms]] |

The technique used in finding the largest term can also be used to estimate the error in using the first p terms in the expansion of $(1+x)^n$ to approximate the value of $(1+x)^n$ for small values of x [--that is, in exercises like those of Part C on page 8-199].

The error is

$$C(n, p)x^p + C(n, p+1)x^{p+1} + \dots + C(n, n)x^n,$$

and its absolute value is, by Theorem 169c, at most

$$(*) \quad |C(n, p)x^p| + |C(n, p+1)x^{p+1}| + \dots + |C(n, n)x^n|.$$

Now, as in the preceding COMMENTARY,

$$\begin{aligned} \frac{|C(n, p+k)x^{p+k}|}{|C(n, p+(k-1))x^{p+(k-1)}|} &= \frac{n - [p+(k-1)]}{[p+(k-1)]+1} |x| \\ &\leq \frac{n-p}{p+1} |x|, \text{ for } k \geq 1. \end{aligned}$$

So, the terms of (*) are at most equal to the corresponding terms of the geometric progression whose first term is $|C(n, p)x^p|$ and whose common ratio is $\frac{n-p}{p+1} |x|$. Since the sum of all the terms of this progression is

$$\frac{|C(n, p)x^p|}{1 - \frac{n-p}{p+1} |x|},$$

it follows that (*) is less than this number. That is, the error after p terms is less than

$$\frac{p+1}{(p+1) - (n-p)|x|} \cdot [\text{the absolute value of the } (p+1)\text{th term}].$$

Applying this result to the computation given in the answer for Exercise 1 of Part C on page 8-199, we see that the sum of the first 5 terms, 1.0706608, differs from $(1.02)^{27}$ by at most

$$\frac{5+1}{(5+1) - (27-5)|0.02|} \cdot 0.000258336.$$

Since this is less than 0.000277, it follows that

$$1.706608 < 1.02^{27} < 1.706855.$$

line 1 on page 8-201: By the recursive definition of C on page 8-168,
for $p - 2 \geq 0$,

$$C(n, p - 1) = C(n, p - 2) \frac{n - (p - 2)}{(p - 2) + 1}.$$

Since, for $0 \leq p - 2 \leq n$, $C(n, p - 2) \neq 0$, it follows that, for $2 \leq p \leq n + 2$,

$$\frac{C(n, p - 1)}{C(n, p - 2)} = \frac{n - (p - 2)}{(p - 2) + 1}.$$

So, in particular, this is the case for $1 < p \leq n + 1$. [The case $p = n + 2$ is of no present interest.] Now, by Exercises 4 and 3 of Part E on page 8-145, it follows that

$$\begin{aligned} \left| \frac{C(n, p - 1) x^{p-1}}{C(n, p - 2) x^{p-2}} \right| &= \left| \frac{C(n, p - 1)}{C(n, p - 2)} \frac{x^{p-1}}{x^{p-2}} \right| \\ &= \left| \frac{C(n, p - 1)}{C(n, p - 2)} \right| \cdot \left| \frac{x^{p-1}}{x^{p-2}} \right| \\ &= \frac{n - (p - 2)}{(p - 2) + 1} |x|. \end{aligned}$$

line 8b on page 8-201: By Theorem 118c,

$$\left[\left| \frac{(n+2)|x| + 1}{1 + |x|} \right| \right] \leq n + 1 \iff \frac{(n+2)|x| + 1}{1 + |x|} < n + 2.$$

Simplifying, we see that this is the case if and only if $1 < n + 2$.
Since $n \geq 1$ and $3 > 1$, this is always the case.

*

Answers for Part ☆G.

1. the 7th term $[= C(18, 6)/2^6 = 4641/16]$
2. the 2nd term $[= C(97, 1) \cdot 0.02 = 1.94]$
3. the 10th term $[= C(15, 9) \cdot 3^6 \cdot 4^9 = 2^{18} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11 \cdot 13]$
4. the 8th term $[= C(256, 7) \cdot (-0.03)^7 = -3^7 \cdot C(256, 7) \cdot 10^{-14}$
 $= -278.85044227104]$

*

$$\begin{aligned}
(d) \quad \sum_{p=1}^n p^3 &= \frac{1}{3+1} \left(n^{3+1} + \sum_{k=1}^3 \left[(-1)^{k+1} C(3+1, k+1) \sum_{p=1}^n p^{3-k} \right] \right) \\
&= \frac{1}{4} \left(n^4 + C(4, 2) \sum_{p=1}^n p^2 - C(4, 3) \sum_{p=1}^n p + C(4, 4) \sum_{p=1}^n 1 \right) \\
&= \frac{1}{4} \left(n^4 + 6 \frac{n(n+1)(2n+1)}{6} - 4 \frac{n(n+1)}{2} + n \right) = \frac{n^2(n+1)^2}{4}
\end{aligned}$$

2. The summation theorem for fourth powers can be obtained by continuing as in the answers for Exercise 1. The result is:

$$\sum_{p=1}^n p^4 = \frac{1}{30} n(n+1)(2n+1)(3n^2 + 3n - 1)$$

Using this and Theorem 131 one obtains, from (*):

$$\begin{aligned}
\sum_{p=1}^n p^5 &= \frac{1}{6} \left(n^6 + 15 \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30} - 20 \frac{n^2(n+1)^2}{4} \right. \\
&\quad \left. + 15 \frac{n(n+1)(2n+1)}{6} - 6 \frac{n(n+1)}{2} + n \right)
\end{aligned}$$

$$\begin{aligned}
3. \quad \sum_{p=1}^n p^5 &= \frac{1}{12} \left(2n^6 + n(n+1)(2n+1)(3n^2 + 3n - 1) - 10n^2(n+1)^2 \right. \\
&\quad \left. + 5n(n+1)(2n+1) - 6n(n+1) + 2n \right) \\
&= \frac{1}{12} \left(n(n+1)[(2n+1)(3n^2 + 3n - 1) + 5(2n+1) - 6] + 2(n^6 + n) - 10n^2(n+1)^2 \right) \\
&= \frac{1}{12} [n(n+1)(2n^4 + 4n^3 + 11n^2 + 9n) - 10n^2(n+1)^2] \\
&= \frac{1}{12} [n(n+1)n(n+1)(2n^2 + 2n + 9) - 10n^2(n+1)^2] \\
&= \frac{1}{12} n^2(n+1)^2(2n^2 + 2n - 1)
\end{aligned}$$

4. [See discussion beginning on page TC[8-203, 204]a.]

*

Answers for ☆Exercises.

$$1. (a) \sum_{p=1}^n 1 = \sum_{p=1}^n p^0 = \frac{1}{0+1} \left(n^{0+1} + \sum_{k=1}^0 \left[(-1)^{k+1} C(0+1, k+1) \sum_{p=1}^n p^{0-k} \right] \right) \\ = \frac{1}{0+1} (n^{0+1} + 0) = n$$

$$(b) \sum_{p=1}^n p = \sum_{p=1}^n p^1 = \frac{1}{1+1} \left(n^{1+1} + \sum_{k=1}^1 \left[(-1)^{k+1} C(1+1, k+1) \sum_{p=1}^n p^{1-k} \right] \right) \\ = \frac{1}{1+1} \left(n^{1+1} + (-1)^{1+1} C(1+1, 1+1) \sum_{p=1}^n p^{1-1} \right) \\ = \frac{1}{2} \left(n^2 + \sum_{p=1}^n 1 \right) = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}$$

$$(c) \sum_{p=1}^n p^2 = \frac{1}{2+1} \left(n^{2+1} + \sum_{k=1}^2 \left[(-1)^{k+1} C(2+1, k+1) \sum_{p=1}^n p^{2-k} \right] \right) \\ = \frac{1}{2+1} \left(n^{2+1} + C(2+1, 1+1) \sum_{p=1}^n p^{2-1} - C(2+1, 2+1) \sum_{p=1}^n p^{2-2} \right) \\ = \frac{1}{3} (n^3 + 3 \frac{n(n+1)}{2} - 1n) = \frac{n(n+1)(2n+1)}{6}$$

Also, (ff) can be proved by induction:

$$(i) \quad \sum_{p=1}^1 \left[\sum_{q=1}^m a_{p,q} \right] = \sum_{q=1}^m a_{1,q} = \sum_{q=1}^m \left[\sum_{p=1}^1 a_{p,q} \right]$$

$$(ii) \quad \text{Suppose that } \sum_{p=1}^n \left[\sum_{q=1}^m a_{p,q} \right] = \sum_{q=1}^m \left[\sum_{p=1}^n a_{p,q} \right]. \quad \text{Then,}$$

$$\begin{aligned} \sum_{p=1}^{n+1} \left[\sum_{q=1}^m a_{p,q} \right] &= \sum_{p=1}^n \left[\sum_{q=1}^m a_{p,q} \right] + \sum_{q=1}^m a_{n+1,q} \quad \left. \vphantom{\sum_{p=1}^{n+1}} \right\} \text{inductive hypothesis} \\ &= \sum_{q=1}^m \left[\sum_{p=1}^n a_{p,q} \right] + \sum_{q=1}^m a_{n+1,q} \quad \left. \vphantom{\sum_{p=1}^{n+1}} \right\} \text{Theorem 134} \\ &= \sum_{q=1}^m \left[\sum_{p=1}^n a_{p,q} + a_{n+1,q} \right] \\ &= \sum_{q=1}^m \left[\sum_{p=1}^{n+1} a_{p,q} \right] \quad \left. \vphantom{\sum_{p=1}^{n+1}} \right\} \text{recursive definition} \end{aligned}$$

Etc.

*

The passage from (ii) to the next line on page 8-204 is by Theorem 136 and the fact that $(-1)^1 C(j+1, 1) = -(j+1)$ [Exercise 1 of Part C on page 8-169].

*

The recursion formula (*) on page 8-204 can be transformed further. For a reference, see the COMMENTARY for page 8-24.

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| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
| 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 |
| 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |

and, by Theorem 133,

$$1 \sum_{p=1}^n (p+1), 2 \sum_{p=1}^n (p+2), 3 \sum_{p=1}^n (p+3), \dots (j+1) \sum_{p=1}^n [p+(j+1)].$$

Adding these, we see that the sum of all the values listed is

$$\sum_{k=1}^{j+1} \left[k \sum_{p=1}^n (p+k) \right].$$

On the other hand, if we return to the table and begin by adding the terms listed in each single row, we obtain, respectively,

$$\sum_{k=1}^{j+1} k(1+k), \sum_{k=1}^{j+1} k(2+k), \sum_{k=1}^{j+1} k(3+k), \dots, \sum_{k=1}^{j+1} k(n+k).$$

Adding these sums we see that the sum of all the values listed is

$$\sum_{p=1}^n \left[\sum_{k=1}^{j+1} k(p+k) \right].$$

Consequently, (f).

Aside from the use of Theorem 133, all that is needed for a rigorous proof is one new theorem:

$$(ff) \quad \forall_m \forall_n \sum_{p=1}^n \left[\sum_{q=1}^m a_{p,q} \right] = \sum_{q=1}^m \left[\sum_{p=1}^n a_{p,q} \right]$$

An intuitive justification of (ff) can be carried out along the same lines as that of (f)--the table looks like:

$$\begin{array}{cccc} a_{1,1} & a_{1,2} & a_{1,3} & \dots a_{1,m} \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots a_{2,m} \\ a_{3,1} & a_{3,2} & a_{3,3} & \dots a_{3,m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots a_{n,m} \end{array}$$

... and will be

$$(\dots)$$

... for ...

...

... to ...

... (consider ...)

$$\dots$$

... This ...

$$\begin{matrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{matrix}$$

... (consider ...)

$$\dots$$

last line on page 8-203: By Theorem 138,

$$\sum_{p=1}^n [p^{j+1} - (p-1)^{j+1}] = n^{j+1} - 0^{j+1}.$$

Now, for $j+1 > 0$ --that is, for $j \geq 0$ -- $0^{j+1} = 0$. So, (i) on page 8-204 follows [for $j \geq 0$].

*

The passage from (i), on page 8-204, to (ii) may or may not trouble your students. If it doesn't, fine [for the present]. If it does, the explanation [which is the answer for Exercise 4 at the foot of page 8-204] is as follows:

Consider [to save writing] some simpler expression than $'(-1)^k C(j+1, k)p^{j+1-k}$, say $'k(p+k)'$. What we need to show is that

$$(f) \quad \sum_{p=1}^n \left[\sum_{k=1}^{j+1} k(p+k) \right] = \sum_{k=1}^n \left[k \sum_{p=1}^n (p+k) \right].$$

The values of $'k(p+k)'$ which are of interest can be arranged in a table:

| | $[k=1]$ | $[k=2]$ | $[k=3] \dots [k=j+1]$ |
|----------|----------|----------|-------------------------------|
| $[p=1]$ | $1(1+1)$ | $2(1+2)$ | $3(1+3) \dots (j+1)[1+(j+1)]$ |
| $[p=2]$ | $1(2+1)$ | $2(2+2)$ | $3(2+3) \dots (j+1)[2+(j+1)]$ |
| $[p=3]$ | $1(3+1)$ | $2(3+2)$ | $3(3+3) \dots (j+1)[3+(j+1)]$ |
| \vdots | \vdots | \vdots | \vdots |
| $[p=n]$ | $1(n+1)$ | $2(n+2)$ | $3(n+3) \dots (j+1)[n+(j+1)]$ |

Adding the terms listed in each single columns we obtain, respectively,

$$\sum_{p=1}^n 1(p+1), \sum_{p=1}^n 2(p+2), \sum_{p=1}^n 3(p+3), \dots, \sum_{p=1}^n (j+1)[p+j+1],$$

$\frac{d}{dt} \left(\frac{1}{\rho} \right) = - \frac{1}{\rho^2} \frac{d\rho}{dt}$

4. *Chlorophyll a* and *Chlorophyll b* contents were determined by the method of Arar and Cook (1987).

1. *Chlorophyll a* (Chl a) and *Chlorophyll b* (Chl b) are the primary photosynthetic pigments in green plants. They are responsible for capturing light energy and converting it into chemical energy through the process of photosynthesis. Chl a is the most abundant pigment, while Chl b is present in smaller amounts. Both pigments absorb light most efficiently in the blue and red regions of the visible spectrum.

1. 25-105

| | | | | | | |
|----|---------------|---|----|-----|-----|----|
| 9. | a: | 2 | 3 | 7 | 9 | -4 |
| | $\Delta a:$ | 1 | 4 | 2 | -13 | |
| | $\Delta^2 a:$ | | 3 | -2 | -15 | |
| | $\Delta^3 a:$ | | -5 | -13 | | |
| | $\Delta^4 a:$ | | | -8 | | |

Since $\Delta^4 a$ is a constant,

$$\sum_{p=1}^n a_p = 2n + 1 \frac{n(n-1)}{2} + 3 \frac{n(n-1)(n-2)}{6} - 5 \frac{n(n-1)(n-2)(n-3)}{24} - 8 \frac{n(n-1)(n-2)(n-3)(n-4)}{120}$$

$$= -\frac{n}{120} (8n^4 - 55n^3 + 70n^2 - 5n - 258).$$

2-24-20

118

1

1000

$$7. \quad a: \quad 2 \quad 20 \quad 72$$

$$\Delta a: \quad 18 \quad 52$$

$$\Delta^2 a: \quad 34$$

$$\Delta^3 a: \quad 3 \cdot 3! = 0 \quad [\text{by the theorems referred to between Exercises 5 and 6}]$$

$$\begin{aligned} \forall_n \sum_{p=1}^n (3p^3 - p^2) &= na_1 + \sum_{k=1}^3 C(n, k+1)(\Delta^k a)_1 \\ &= n \cdot 2 + C(n, 2) \cdot 18 + C(n, 3) \cdot 34 + C(n, 4) \cdot 18 \\ &= 2n + 18 \frac{n(n-1)}{2} + 34 \frac{n(n-1)(n-2)}{6} \\ &\quad + 18 \frac{n(n-1)(n-2)(n-3)}{24} \\ &= \frac{1}{12} [24n + 108n(n-1) + 68n(n-1)(n-2) \\ &\quad + 9n(n-1)(n-2)(n-3)] \\ &= \frac{n}{12} [9n^3 + 14n^2 + 3n - 2] \\ &= \frac{1}{12} n(n+1)(9n^2 + 5n - 2) \end{aligned}$$

$$8. \quad a: \quad 5 \quad 13$$

$$\Delta a: \quad 8$$

$$\Delta^2 a: \quad -2 \cdot 2 \quad [\text{since } a_p = -2p^2 + \dots]$$

$$\begin{aligned} \sum_{p=1}^n a_p &= 5n + 8 \frac{n(n-1)}{2} - 4 \frac{n(n-1)(n-2)}{6} \\ &= -\frac{n}{3} (2n^2 - 18n + 1) \end{aligned}$$

The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \int_0^x \frac{1}{1+t^2} dt$$
 for $x \in \mathbb{R}$. It is shown that $f(x)$ is an odd function and that $f(x) \in C^1(\mathbb{R})$. Moreover, it is proved that $f(x)$ is a strictly increasing function and that $f(x) \in C^2(\mathbb{R})$.

In the second part of the paper, we study the properties of the function $g(x)$ defined by the equation

$$g(x) = \int_0^x \frac{t}{1+t^2} dt$$
 for $x \in \mathbb{R}$. It is shown that $g(x)$ is an even function and that $g(x) \in C^1(\mathbb{R})$. Moreover, it is proved that $g(x)$ is a strictly increasing function and that $g(x) \in C^2(\mathbb{R})$.

In the third part of the paper, we study the properties of the function $h(x)$ defined by the equation

$$h(x) = \int_0^x \frac{t^2}{1+t^2} dt$$
 for $x \in \mathbb{R}$. It is shown that $h(x)$ is an odd function and that $h(x) \in C^1(\mathbb{R})$. Moreover, it is proved that $h(x)$ is a strictly increasing function and that $h(x) \in C^2(\mathbb{R})$.

In the fourth part of the paper, we study the properties of the function $k(x)$ defined by the equation

$$k(x) = \int_0^x \frac{t^3}{1+t^2} dt$$
 for $x \in \mathbb{R}$. It is shown that $k(x)$ is an even function and that $k(x) \in C^1(\mathbb{R})$. Moreover, it is proved that $k(x)$ is a strictly increasing function and that $k(x) \in C^2(\mathbb{R})$.

In the fifth part of the paper, we study the properties of the function $l(x)$ defined by the equation

$$l(x) = \int_0^x \frac{t^4}{1+t^2} dt$$
 for $x \in \mathbb{R}$. It is shown that $l(x)$ is an odd function and that $l(x) \in C^1(\mathbb{R})$. Moreover, it is proved that $l(x)$ is a strictly increasing function and that $l(x) \in C^2(\mathbb{R})$.

In the sixth part of the paper, we study the properties of the function $m(x)$ defined by the equation

$$m(x) = \int_0^x \frac{t^5}{1+t^2} dt$$
 for $x \in \mathbb{R}$. It is shown that $m(x)$ is an even function and that $m(x) \in C^1(\mathbb{R})$. Moreover, it is proved that $m(x)$ is a strictly increasing function and that $m(x) \in C^2(\mathbb{R})$.

In the seventh part of the paper, we study the properties of the function $n(x)$ defined by the equation

$$n(x) = \int_0^x \frac{t^6}{1+t^2} dt$$
 for $x \in \mathbb{R}$. It is shown that $n(x)$ is an odd function and that $n(x) \in C^1(\mathbb{R})$. Moreover, it is proved that $n(x)$ is a strictly increasing function and that $n(x) \in C^2(\mathbb{R})$.

In the eighth part of the paper, we study the properties of the function $o(x)$ defined by the equation

$$o(x) = \int_0^x \frac{t^7}{1+t^2} dt$$
 for $x \in \mathbb{R}$. It is shown that $o(x)$ is an even function and that $o(x) \in C^1(\mathbb{R})$. Moreover, it is proved that $o(x)$ is a strictly increasing function and that $o(x) \in C^2(\mathbb{R})$.

In the ninth part of the paper, we study the properties of the function $p(x)$ defined by the equation

$$p(x) = \int_0^x \frac{t^8}{1+t^2} dt$$
 for $x \in \mathbb{R}$. It is shown that $p(x)$ is an odd function and that $p(x) \in C^1(\mathbb{R})$. Moreover, it is proved that $p(x)$ is a strictly increasing function and that $p(x) \in C^2(\mathbb{R})$.

The proofs of the other two theorems mentioned are trivial:

By definition, $\forall_p (a + b)_p = a_p + b_p$. By definition, $\forall_n [\Delta(a + b)]_n = (a + b)_{n+1} - (a + b)_n$. So, $[\Delta(a + b)]_n = (a_{n+1} + b_{n+1}) - (a_n + b_n) = (a_{n+1} - a_n) + (b_{n+1} - b_n) = (\Delta a)_n + (\Delta b)_n$.

By definition, $\forall_p (ab)_p = a_p b_p$. So, $[\Delta(ab)]_n = a_{n+1} b_{n+1} - a_n b_n$. But, if a is a constant sequence, $a_{n+1} = a_n$, and, so, $a_{n+1} b_{n+1} - a_n b_n = a_n b_{n+1} - a_n b_n = a_n (b_{n+1} - b_n) = a_n (\Delta b)_n$.

*

| | | | | | | |
|----|----------------|----|-----|-----|------|------|
| 6. | a: | 1 | 32 | 243 | 1024 | 3125 |
| | Δa : | 31 | 211 | 781 | 2101 | |
| | $\Delta^2 a$: | | 180 | 570 | 1320 | |
| | $\Delta^3 a$: | | | 390 | 750 | |
| | $\Delta^4 a$: | | | | 360 | |

Since, for each p , $(\Delta^5 a)_p = 5! = 120$, it follows from Theorem 180b that

$$\begin{aligned}
 \forall_n \sum_{p=1}^n p^5 &= na_1 + \sum_{k=1}^n C(n, k+1)(\Delta^k a)_1 \\
 &= n \cdot 1 + C(n, 2) \cdot 31 + C(n, 3) \cdot 180 + C(n, 4) \cdot 390 \\
 &\quad + C(n, 5) \cdot 360 + C(n, 6) \cdot 120 \\
 &= n + 31 \frac{n(n-1)}{2} + 180 \frac{n(n-1)(n-2)}{6} + 390 \frac{n(n-1)(n-2)(n-3)}{24} \\
 &\quad + 360 \frac{n(n-1)(n-2)(n-3)(n-4)}{120} \\
 &\quad + 120 \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{720}
 \end{aligned}$$

$[= \frac{1}{12} n^2 (n+1)^2 (2n^2 + 2n - 1)]$, as in Exercise 3 on page 8-204].

1. The first part of the paper is devoted to the study of the

properties of the function $f(x)$ defined by the equation

$$f(x) = \int_0^x \frac{1}{1+t^2} dt$$

It is shown that the function $f(x)$ is continuous and differentiable

for all values of x . The derivative of the function is given by the

$$f'(x) = \frac{1}{1+x^2}$$

It is also shown that the function $f(x)$ is bounded for all values of x .

2. In the second part of the paper, we consider the function

$$g(x) = \int_0^x \frac{t}{1+t^2} dt$$

It is shown that the function $g(x)$ is continuous and differentiable

for all values of x . The derivative of the function is given by the

$$g'(x) = \frac{x}{1+x^2}$$

It is also shown that the function $g(x)$ is bounded for all values of x .

5. Suppose that b is a sequence when m th difference sequence is a

constant. By Theorem 180a, $\forall_p b_p = b_1 + \sum_{k=1}^m C(p-1, k)(\Delta^k b)_1$.

Hence,

$$\begin{aligned} \sum_{p=1}^n b_p &= \sum_{p=1}^n \left[b_1 + \sum_{k=1}^m C(p-1, k)(\Delta^k b)_1 \right] \\ &= \sum_{p=1}^n b_1 + \sum_{p=1}^n \left[\sum_{k=1}^m C(p-1, k)(\Delta^k b)_1 \right] \\ &= \sum_{p=1}^n b_1 + \sum_{k=1}^m \left[(\Delta^k b)_1 \cdot \sum_{p=1}^n C(p-1, k) \right] \\ &= nb_1 + \sum_{k=1}^m \left[(\Delta^k b)_1 \cdot C(n, k+1) \right] \\ &= nb_1 + \sum_{k=1}^m C(n, k+1)(\Delta^k b)_1. \end{aligned}$$

So, for each sequence a whose m th difference-sequence is a constant,

$$\forall_n \sum_{p=1}^n a_p = na_1 + \sum_{k=1}^m C(n, k+1)(\Delta^k a)_1.$$

*

The proof that, if a is the sequence of m th powers then $\Delta^m a$ is the constant sequence whose value is $m!$ is difficult [on the basis of the results already at hand]. However, students have a little evidence for believing this from Part C on page 8-63.

1. The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \int_0^x \frac{1}{1+t^2} dt, \quad x \in \mathbb{R}.$$

It is shown that the function $f(x)$ is strictly increasing and concave down on the interval $(-\infty, \infty)$.

2. In the second part, we consider the function $g(x)$ defined by the equation

$$g(x) = \int_0^x \frac{1}{1+t^4} dt, \quad x \in \mathbb{R}.$$

It is shown that the function $g(x)$ is strictly increasing and concave down on the interval $(-\infty, \infty)$.

3. The third part of the paper is devoted to the study of the function $h(x)$ defined by the equation

$$h(x) = \int_0^x \frac{1}{1+t^6} dt, \quad x \in \mathbb{R}.$$

It is shown that the function $h(x)$ is strictly increasing and concave down on the interval $(-\infty, \infty)$.

4. In the fourth part, we consider the function $k(x)$ defined by the equation

$$k(x) = \int_0^x \frac{1}{1+t^8} dt, \quad x \in \mathbb{R}.$$

It is shown that the function $k(x)$ is strictly increasing and concave down on the interval $(-\infty, \infty)$.

5. The fifth part of the paper is devoted to the study of the function $l(x)$ defined by the equation

$$l(x) = \int_0^x \frac{1}{1+t^{10}} dt, \quad x \in \mathbb{R}.$$

It is shown that the function $l(x)$ is strictly increasing and concave down on the interval $(-\infty, \infty)$.

6. In the sixth part, we consider the function $m(x)$ defined by the equation

$$m(x) = \int_0^x \frac{1}{1+t^{12}} dt, \quad x \in \mathbb{R}.$$

It is shown that the function $m(x)$ is strictly increasing and concave down on the interval $(-\infty, \infty)$.

7. The seventh part of the paper is devoted to the study of the function $n(x)$ defined by the equation

$$n(x) = \int_0^x \frac{1}{1+t^{14}} dt, \quad x \in \mathbb{R}.$$

It is shown that the function $n(x)$ is strictly increasing and concave down on the interval $(-\infty, \infty)$.

8. In the eighth part, we consider the function $o(x)$ defined by the equation

$$o(x) = \int_0^x \frac{1}{1+t^{16}} dt, \quad x \in \mathbb{R}.$$

It is shown that the function $o(x)$ is strictly increasing and concave down on the interval $(-\infty, \infty)$.

(ii) Suppose that, for each sequence a whose m th difference-sequence

is a constant, $\forall_n a_n = a_1 + \sum_{k=1}^m C(n-1, k)(\Delta^k a)_1$. [This is the inductive

hypothesis.] Suppose that b is a sequence such that $\Delta^{m+1}b$ is a constant. Since $\Delta^{m+1}b = \Delta^m(\Delta b)$, Δb is a sequence whose m th difference-sequence is a constant. So, by the inductive hypothesis,

$$\begin{aligned}\forall_n (\Delta b)_n &= (\Delta b)_1 + \sum_{k=1}^m C(n-1, k)(\Delta^k \Delta b)_1 = (\Delta b)_1 + \sum_{k=1}^m C(n-1, k)(\Delta^{k+1}b)_1 \\ &= (\Delta b)_1 + \sum_{k=1}^{m+1} C(n-1, k-1)(\Delta^k b)_1 = (\Delta b)_1 + \sum_{k=2}^{m+1} C(n-1, k-1)(\Delta^k b)_1.\end{aligned}$$

$$\text{By Theorem 140, } b_q = b_1 + \sum_{p=1}^{q-1} (\Delta b)_p.$$

So,

$$\begin{aligned}b_q &= b_1 + \sum_{p=1}^{q-1} \left[\sum_{k=1}^{m+1} C(p-1, k-1)(\Delta^k b)_1 \right] \\ &= b_1 + \sum_{k=1}^{m+1} \left[(\Delta^k b)_1 \sum_{p=1}^{q-1} C(p-1, k-1) \right] \\ &= b_1 + \sum_{k=1}^{m+1} [(\Delta^k b)_1 \circ C(q-1, k)] \\ &= b_1 + \sum_{k=1}^{m+1} C(q-1, k)(\Delta^k b)_1.\end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{Theorem 179a}$$

Hence, for each sequence a whose $(m+1)$ th difference-sequence is

a constant, $\forall_n a_n = a_1 + \sum_{k=1}^{m+1} C(n-1, k)(\Delta^k a)_1$.

Etc.

Let \mathcal{H} be a Hilbert space and \mathcal{K} be a closed subspace of \mathcal{H} . Then $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$.

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3. Theorem 132

[The instances of Theorem 179b for $k = 0, 1, 2$, and 3 are:

$$\sum_{p=1}^n C(p-1, 0) = C(n, 1) \quad \text{-- that is: } \sum_{p=1}^n 1 = n$$

$$\sum_{p=1}^n C(p, 1) = C(n+1, 2) \quad \text{-- that is: } \sum_{p=1}^n p = \frac{(n+1)n}{2}$$

$$\sum_{p=1}^n C(p+1, 2) = C(n+2, 3) \quad \text{-- that is: } \sum_{p=1}^n \frac{(p+1)p}{2} = \frac{(n+2)(n+1)n}{6}$$

$$\sum_{p=1}^n C(p+2, 3) = C(n+3, 4) \quad \text{-- that is: } \sum_{p=1}^n \frac{(p+2)(p+1)p}{6} = \frac{(n+3)(n+2)(n+1)n}{24}$$

These are equivalent to the four parts of Theorem 132.]

[Exercises 4 through 9 are on page 8-206.]

4. [As suggested in the text following the statement of Theorem 180, we use induction on 'm'. In the following proof we use 'b' as a test-pattern variable.]

(i) By Theorem 140, $b_q = b_1 + \sum_{p=1}^{q-1} (\Delta b)_p$. Suppose that Δb is a constant. Then, by Theorems 133 and 131a, $\sum_{p=1}^{q-1} (\Delta b)_p = (q-1)(\Delta b)_1$

$= \sum_{k=1}^1 C(q-1, k)(\Delta^k b)_1$. Hence, for each sequence a whose 1st difference-sequence is a constant, $\forall_n a_n = a_1 + \sum_{k=1}^1 C(q-1, k)(\Delta^k a)_1$.

Let \mathcal{A} be a subalgebra of \mathcal{B} and let \mathcal{C} be a subalgebra of \mathcal{A} .

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B) \quad \text{if } A \text{ and } B \text{ are independent.}$$

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Let \mathcal{A} be a subalgebra of \mathcal{B} and let \mathcal{C} be a subalgebra of \mathcal{A} . Then \mathcal{C} is a subalgebra of \mathcal{B} .

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$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B) \quad \text{if } A \text{ and } B \text{ are independent.}$$

Let \mathcal{A} be a subalgebra of \mathcal{B} and let \mathcal{C} be a subalgebra of \mathcal{A} . Then \mathcal{C} is a subalgebra of \mathcal{B} .

Let

Let \mathcal{A} be a subalgebra of \mathcal{B} and let \mathcal{C} be a subalgebra of \mathcal{A} .

Let \mathcal{A} be a subalgebra of \mathcal{B} and let \mathcal{C} be a subalgebra of \mathcal{A} . Then \mathcal{C} is a subalgebra of \mathcal{B} .

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Let \mathcal{A} be a subalgebra of \mathcal{B} and let \mathcal{C} be a subalgebra of \mathcal{A} .

line 9 on page 8-205: Here are the rewritten parts of Theorem 139:

$$(a) \forall_n \sum_{p=1}^n C(p-1, 1) = C(n, 2)$$

$$(b) \forall_n \sum_{p=1}^n C(p-1, 2) = C(n, 3)$$

$$(c) \forall_n \sum_{p=1}^n C(p-1, 3) = C(n, 4)$$

$$(d) \forall_n \sum_{p=1}^n C(p-1, 4) = C(n, 5)$$

In obtaining (b), one "divides Theorem 139 through by 2" and applies Theorem 171. In obtaining (c), one "divides Theorem 139c through by 6" and applies Theorem 171. Etc.

line 15: The instance of Theorem 179a for $m = 1$ is:

$$\forall_n \sum_{p=1}^n C(p-1, 0) = C(n, 1)$$

and is equivalent [by Theorem 171--or by the recursive definition of C on page 8-168 and Exercise 1 of Part C on page 8-169] to Theorem 131a [or: Theorem 132a].

*

Answers for ☆Exercises.

1. $[b_1 = a_1:]$ For $m > 1$, $C(1, m) = 0 = C(1-1, m-1)$;
and $C(1, 1) = 1 = C(1-1, 1-1)$.

$[\forall_n b_{n+1} = b_n + a_{n+1}:]$ By Theorem 173, $\forall_n C(n+1, m)$
 $= C(n, m) + C(n, m-1)$.

Consequently, by Theorem 130, Theorem 179a.

- 2.. $C(1+k, k+1) = 1 = C(k+1-1, k)$ and, by Theorem 173,
 $\forall_n C(n+1+k, k+1) = C(n+k, k+1) + C(n+k, k)$. Consequently, by
Theorem 130, Theorem 179b.

The material on pages 8-207 through 8-217 is, as indicated, optional. It may be thought of as a continuation of related material in Unit 4 [starting with page 4-49] and in Unit 7 [starting with page 7-115]. However, the treatment here does not presuppose detailed knowledge of this earlier material.

You may wish to discuss pages 8-207 and 8-208 with your students --possibly omitting the proof of Theorem 181--and to point out that Theorem 183 on page 8-213 is (7) in the list, on page 7-47, of the assumptions which were made in Unit 4 concerning the positive integers. With the proof of this theorem [which begins somewhat below the middle of page 8-215] all seven of these assumptions have been shown to be consequences of our basic principles.

The short bibliography on page 8-217 should also be called to your students' attention.

Answers for Review Exercises.

1. (a) none $[5p + 3 = 3p - 1 \iff p = -2, \text{ and } -2 \notin \mathbb{I}^+]$

★(b) $a_m = b_n \iff \exists_{k \geq 0} (m = 3k + 1 \text{ and } n = 5k + 3)$

[So, for examples, $a_1 = b_3$ and $a_4 = b_8$. In fact, $a_m = b_n$

$\iff 5m + 3 = 3n - 1$, --that is, $\iff 5m - 3n = -4$. Since, by inspection, $5 \cdot 1 - 3 \cdot 3 = -4$, it follows that $5m - 3n = -4$

$\iff 5(m - 1) - 3(n - 3) = 0$. But, since 5 and 3 are relatively prime, this is the case if and only if, for some k , $m - 1 = 3k$ and $n - 3 = 5k$. Moreover, $3k + 1 > 0$ and $5k + 3 > 0$ if and only if $k \geq 0$.]

2. 63, 80, 99 $[(n - 2)n + (n - 1)(n + 1) + n(n + 2) = 242 \iff 3n^2 - 1 = 242$
--that is, $\iff n = 9]$

3. (a) 4 $[\text{By definition, } \sum_{p=3}^4 (2p - 5) = (2 \cdot 3 - 5) + 2 \cdot 4 - 5), \text{ and}$
since, for each $p > 3$, $2p - 5 > 0$, there is no other solution.]

(b) 6 $[\text{By definition, } \sum_{p=7}^6 (9p^2 + 2p + 4) = 0, \text{ and since, for each}$
 $p \geq 7$, $9p^2 + 2p + 4 > 0$, there is no other solution.]

(c) 1, 2, 3, ..., 10 [or: the solution-set is $\{n : n \leq 10\}$]

(d) 10 $[3 \frac{(n+4)(n+5)}{2} = 315 \iff n = 10]$

(e) 1, 2, 3, ..., 16 [We are to solve ' $\frac{n(n+1)}{2} < 153$ '. Since $17^2 < 306 < 18^2$, $16 \cdot 17 < 306$ and $18 \cdot 19 > 306$. So, each positive integer which is less than or equal to 16 is a

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solution, and none greater than or equal to 18 is a solution.

Since $17 \cdot 18 = 306$, 17 is not a solution. Alternatively,

$$n(n+1) < 306 \iff (n+18)(n-17) < 0.]$$

(f) 821, 822, 823, ..., 861 [$820 < n \leq 861$]

(g) 8 [We are to solve $\frac{n(n+1)}{2} < 41 \leq \frac{(n+1)(n+2)}{2}$. This is equivalent to $\sqrt{n(n+1)} < \sqrt{82} < \sqrt{(n+1)(n+2)}$. Since, for each n , $n < \sqrt{n(n+1)}$, and since $\sqrt{82} < 10$, each solution must be < 10 . Similarly, since $9 < \sqrt{82}$ and $\sqrt{(n+1)(n+2)} < n+2$, each solution must be > 7 . So, the only numbers which can be solutions are 8 and 9. Testing these, 8 is a solution and 9 is not.

Alternatively, the sentence to be solved is equivalent to:

$$(*) \quad n^2 + n - 82 < 0 \leq n^2 + 3n - 80$$

Solving the quadratic equations ' $x^2 + x - 82 = 0$ ' and ' $x^2 + 3x - 80 = 0$ ' it can be seen that $x^2 + x - 82 < 0 \iff \frac{-1 - \sqrt{329}}{2} < x < \frac{-1 + \sqrt{329}}{2}$ and that $x^2 + 3x - 80 \geq 0 \iff (x \leq \frac{-3 - \sqrt{329}}{2} \text{ or } \frac{-3 + \sqrt{329}}{2} \leq x)$. So, (*) is equivalent to:

$$n < \frac{-1 + \sqrt{329}}{2} \text{ and } n \geq \frac{-3 + \sqrt{329}}{2}$$

--that is, to ' $8 \leq n < 9$ '.]

$$4. (a) (1) (i) \quad b_1 = \frac{1 \cdot 2 \cdot 3 \cdot 4}{12} = 2; \quad a_1 = 1^2(1+1) = 2$$

$$\begin{aligned} (ii) \quad b_n + a_{n+1} &= \frac{n(n+1)(n+2)(3n+1)}{12} + (n+1)^2(n+2) \\ &= \frac{(n+1)(n+2) [n(n+1) + 12(n+1)]}{12} \\ &= \frac{(n+1)(n+2) [3n^2 + n + 12n + 12]}{12} \\ &= \frac{(n+1)(n+2)(n+3)(3n+4)}{12} = b_{n+1} \end{aligned}$$


$$(2) \text{ (i) } \sum_{p=1}^1 p^2(p+1) = 1^2(1+1) = 2 = \frac{1 \cdot 2 \cdot 3 \cdot 4}{12}$$

$$\begin{aligned} \text{(ii) } \sum_{p=1}^{q+1} p^2(p+1) &= \sum_{p=1}^q p^2(p+1) + (q+1)^2(q+2) \\ &= \frac{q(q+1)(q+2)(3q+1)}{12} + (q+1)^2(q+2) \\ &= \frac{(q+1)(q+2)(q+3)(3q+4)}{12} \end{aligned}$$

$$(b) \quad \forall_p p(p+1)^2 - p^2(p+1) = p(p+1) \quad [\text{algebra}]$$

$$\begin{aligned} \sum_{p=1}^q p(p+1)^2 &= \sum_{p=1}^q p(p+1) + \sum_{p=1}^q p^2(p+1) \\ &= \frac{q(q+1)(q+2)}{3} + \frac{q(q+1)(q+2)(3q+1)}{12} \\ &= \frac{q(q+1)(q+2)(3q+5)}{12} \end{aligned}$$

$$\begin{aligned} \text{(c) } \sum_{p=1}^q (p-1)^2 p &= \sum_{k=0}^{q-1} k^2(k+1) = \sum_{p=1}^{q-1} p^2(p+1) \\ &= \frac{(q-1)q(q+1)[3(q-1)+1]}{12} = \frac{q(q^2-1)(3q-2)}{12} \end{aligned}$$


 [by part (b), above, if $q > 1$; obvious if $q = 1$]

$$p(x) = p(x) + p(x)$$

$$p(x) = p(x) + p(x)$$

$$p(x) = p(x) + p(x)$$

$$p(x) = p(x) + p(x)$$

$$p(x) = p(x) + p(x)$$

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$$p(x) = p(x) + p(x)$$

$$p(x) = p(x) + p(x)$$

$$p(x) = p(x) + p(x)$$

$$\begin{aligned}
(\Delta a)_q &= (\Delta a)_1 + \sum_{p=1}^{q-1} (\Delta^2 a)_p \\
&= [(\Delta a)_2 - \sum_{p=1}^1 (\Delta^2 a)_p] + \sum_{p=1}^{q-1} (\Delta^2 a)_p \\
&= (50 - 22) + \sum_{p=1}^{q-1} [22 + 6(p-1)] \\
&= 28 + 22(q-1) + 6 \frac{(q-1)(q-2)}{2}
\end{aligned}$$

$$\begin{aligned}
a_q &= a_1 + \sum_{p=1}^{q-1} (\Delta a)_p \\
&= 12 + \sum_{p=1}^{q-1} [28 + 22(p-1) + 6 \frac{(p-1)(p-2)}{2}] \\
&= 12 + 28(q-1) + 22 \frac{(q-1)(q-2)}{2} + 6 \frac{(q-1)(q-2)(q-3)}{6} \\
&= 12 + 28(q-1) + 11(q-1)(q-2) + (q-1)(q-2)(q-3)
\end{aligned}$$

$$\begin{aligned}
a_{100} &= 12 + 28 \cdot 99 + 11 \cdot 99 \cdot 98 + 99 \cdot 98 \cdot 97 \\
&= 12 + 2772 + 106722 + 941094 = 1050600
\end{aligned}$$

[illegible]

☆9. There were probably either 35 or 65 marbles in the bag.

[Suppose that there were n marbles in the bag. Since there was 1 left over when John counted the marbles by twos, n is odd. Since there were 2 left over when he counted them by threes, there is an integer $j \geq 0$ such that $n = 3j + 2$. Since n is odd, j is odd--say, $j = 2k + 1$. So, for some integer $k \geq 0$, $n = 6k + 5$. Since n is divisible by 5, so is k . Hence, for some integer $i \geq 0$, $k = 5i$ and $n = 30i + 5$. Consequently, $n = 5$, or 35, or 65, etc. Now, John could scarcely have forgotten the number had he had only 5 marbles--especially since he had counted them by fives! And a small bag would be unlikely to have contained 95 or more marbles. So, the only likely possibilities are 35 and 65.

10. 9825/176851

$$\left[\sum_{p=1}^{100} \frac{1}{p(p+1)(p+2)(p+3)} = \frac{1}{3} \sum_{p=1}^{100} \frac{1}{p(p+1)(p+2)} - \frac{1}{(p+1)(p+2)(p+3)} \right. \\ \left. = \frac{1}{3} \left[\frac{1}{1 \cdot 2 \cdot 3} - \frac{1}{101 \cdot 102 \cdot 103} \right] = \frac{9825}{176851} \right]$$

11. 1050600

$$[(\Delta^2 a)_q = (\Delta^2 a)_1 + \sum_{p=1}^{q-1} (\Delta^3 a)_p \\ = \left[(\Delta^2 a)_3 - \sum_{p=1}^2 (\Delta^3 a)_p \right] + \sum_{p=1}^{q-1} (\Delta^3 a)_p \\ = (34 - 12) + 6(q - 1)]$$

$$\begin{aligned}
 \text{[or: } 10 \sum_{p=1}^{498} (p+2) &= 10 \sum_{p=1}^{498} p + 2 \sum_{p=1}^{498} 1 \\
 &= 10 \frac{498 \cdot 499}{2} + 2 \cdot 498 \\
 &= 10[124251 + 996] = 1252470 \text{]}
 \end{aligned}$$

★6. [See Part B on pages 8-40 and 8-41.]

$$\begin{aligned}
 U_n &= \sum_{p=1}^n \frac{1}{n} \left[\left(\frac{p}{n} \right)^2 + \frac{p}{n} \right] = \frac{1}{n^3} \sum_{p=1}^n p(p+n) \\
 &= \frac{1}{n^3} \cdot \frac{n(n+1)(5n+1)}{6} \\
 &= \frac{(1 + \frac{1}{n})(5 + \frac{1}{n})}{6}
 \end{aligned}$$

$$\begin{aligned}
 L_n &= \sum_{p=1}^{n-1} \frac{1}{n} \left[\left(\frac{p}{n} \right)^2 + \frac{p}{n} \right] = \frac{1}{n^3} \cdot \frac{(n-1)n(5n-1)}{6} \\
 &= \frac{(1 - \frac{1}{n})(5 - \frac{1}{n})}{6}
 \end{aligned}$$

Guess: area-measure = $\frac{5}{6}$

7. $7 \left[\frac{n(n+1)(2n+1)}{6} - n = 19n \iff n(2n^2 + 3n - 119) = 0 \right]$

8. \bar{a} is the sequence b such that, for each p , $b_p = \frac{(p+1)(p+2)}{3}$.

$$[(\bar{a})_n = \frac{1}{n} \sum_{p=1}^n p(p+1) = \frac{1}{n} \cdot \frac{n(n+1)(n+2)}{3} = \frac{(n+1)(n+2)}{3}]$$

1. *Pharmaceutical industry* – The pharmaceutical industry is the largest of the three industries, with sales of \$10.5 billion in 1997. It is the only industry that has not experienced a decline in sales since 1990. The industry is dominated by a few large firms, with the top five firms accounting for 40% of sales. The industry is highly competitive, with many firms competing for market share.

1. *Phragmites australis* (Cav.) Trin. ex Steud.
 2. *Phragmites australis* (Cav.) Trin. ex Steud.
 3. *Phragmites australis* (Cav.) Trin. ex Steud.
 4. *Phragmites australis* (Cav.) Trin. ex Steud.
 5. *Phragmites australis* (Cav.) Trin. ex Steud.
 6. *Phragmites australis* (Cav.) Trin. ex Steud.
 7. *Phragmites australis* (Cav.) Trin. ex Steud.
 8. *Phragmites australis* (Cav.) Trin. ex Steud.
 9. *Phragmites australis* (Cav.) Trin. ex Steud.
 10. *Phragmites australis* (Cav.) Trin. ex Steud.

1. *Chlorophyll a* and *Chlorophyll b* were determined by the method of Lichtenthaler and Sponholz (1980).

$$\begin{aligned}
 5. \quad (a) \quad \sum_{p=1}^{50} (p-1)(p+1) &= \sum_{p=1}^{50} p(p+1) - \sum_{p=1}^{50} p - \sum_{p=1}^{50} 1 \\
 &= \frac{50 \cdot 51 \cdot 52}{3} - \frac{50 \cdot 51}{2} - 50 = 42875
 \end{aligned}$$

$$\begin{aligned}
 [\text{or: } \sum_{p=1}^{50} (p-1)(p+1) &= \sum_{p=1}^{50} (p^2 - 1) = \sum_{p=1}^{50} p^2 - \sum_{p=1}^{50} 1 \\
 &= \frac{50 \cdot 51 \cdot 101}{6} - 50 = 42875 \quad]
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \sum_{p=1}^{60} (p-1)(p+5) &= \sum_{p=1}^{60} (p-1)(p-2) + 7 \sum_{p=1}^{60} (p-1) \\
 &= \frac{60 \cdot 59 \cdot 58}{3} + 7 \cdot \frac{60 \cdot 59}{2} \\
 &= 60 \cdot 59 \left[\frac{58}{3} + \frac{7}{2} \right] = 80830
 \end{aligned}$$

$$\begin{aligned}
 [\text{or: } \sum_{p=1}^{60} (p-1)(p+5) &= \sum_{p=1}^{60} p^2 + 4 \sum_{p=1}^{60} p - 5 \sum_{p=1}^{60} 1 \\
 &= \frac{60 \cdot 61 \cdot 121}{6} + 4 \cdot \frac{60 \cdot 61}{2} - 5 \cdot 60 \\
 &= 73810 + 7320 - 300 = 80830 \quad]
 \end{aligned}$$

$$\begin{aligned}
 (c) \quad 10 \sum_{p=1}^{498} (p+2) &= 10 \sum_{p=3}^{500} p = 10 \left[\frac{500 \cdot 501}{2} - (1+2) \right] \\
 &= 10[125250 - 3] = 1252470
 \end{aligned}$$

$$12. (a) a_n = a_1 + \sum_{p=1}^{n-1} (\Delta a)_p$$

$$= a_1 + \sum_{p=1}^{n-1} d$$

$$= a_1 + (n-1)d$$

$$s_n = \sum_{p=1}^n [a_1 + (p-1)d]$$

$$= a_1 n + d \frac{n(n-1)}{2}$$

$$= \frac{n}{2} [2a_1 + (n-1)d]$$

$$(b) 8 \quad \left[\frac{m}{n} [24 + 4(m-1)] = 208 \iff m(6 + m - 1) = 104 \right. \\ \left. \iff (m-8)(m+13) = 0 \right]$$

$$(c) 8; 59 \quad \left[\frac{7.75 - 0.5}{2 - 31} = \frac{0.5 - c_1}{31 - 1} \right]$$

$$\frac{7.25}{-29} = \frac{0.5 - c_1}{30}$$

$$217.5 = -14.5 + 29c_1$$

$$29c_1 = 232$$

$$c_1 = 8$$

$$\frac{7.75 - 0.5}{2 - 31} = \frac{7.75 + 6.5}{2 - n}$$

$$\frac{7.25}{-29} = \frac{14.25}{2 - n}$$

$$14.5 - 7.25n = -413.25$$

$$7.25n = 427.75$$

$$n = 59 \quad]$$

$$13. -\frac{7}{3} \quad [(3+x)^2 = (1+x)(2+x)]$$

$$14. 7\sqrt{3}; \sqrt{146} \quad \left[\frac{7\sqrt{3} + 1 + 7\sqrt{3} - 1}{2} = 7\sqrt{3}; \sqrt{(7\sqrt{3} + 1)(7\sqrt{3} - 1)} = \sqrt{147 - 1} \right]$$

$$15. (a) \frac{91}{99}$$

$$(b) \frac{71}{110}$$

$$(c) \frac{10847}{3330}$$

$$(d) \frac{11}{150}$$

$$16. (a) \frac{1}{2}$$

$$(b) \frac{4}{9}$$

1911-1912

1912-1913

1913-1914

1914-1915

1915-1916

1916-1917

1917-1918

1918-1919

1919-1920

1920-1921

1921-1922

1922-1923

1923-1924

1924-1925

1925-1926

$$17. \text{ (a) } \frac{my}{m - y}$$

$$\text{ (b) } a^2b^2$$

$$\text{ (c) } \frac{b + 1}{b^3}$$

$$\text{ (d) } \frac{12}{7}$$

$$\text{ (e) } \frac{jk}{(j + k)^2}$$

$$\text{ (f) } 72$$

$$\text{ (g) } \left(\frac{b}{c}\right)^{m-n} [m \geq n]$$

$$\text{ (h) } \frac{x^2}{x - 1}$$

$$\text{ (i) } \frac{6^6y^5z^2}{x^7}$$

$$\text{ (j) } \frac{1}{a^{13}}$$

$$(ii) \quad \left[\sum_{p=1}^{k+2} a_p \right] = (a_{k+2}) \left(\left[\sum_{p=1}^{k+1} a_p \right] \right) \quad [\text{recursive definition}]$$

$$= (a_{k+2}) \left(\left[\sum_{p=1}^k a_{p+1} \right] \right) \quad [\text{inductive hypothesis}]$$

$$= \left(a_{(k+1)+1} \right) \left(\left[\sum_{p=1}^k a_{p+1} \right] \right)$$

$$= \left[\sum_{p=1}^{k+1} a_{p+1} \right] \quad [\text{recursive definition}]$$

[Part (c) of Exercise 20, and its solution, may be made clearer by introducing the sequence b such that, for each p , $b_p = a_{p+1}$. The problem then, is to prove:

$$a_1 = 1 \Rightarrow \quad k \geq 0 \quad \left[\sum_{p=1}^{k+1} a_p \right] = \left[\sum_{p=1}^k b_p \right]$$

Now, in the solution, replace ' a_{p+1} ' (in three places) by ' b_p ', and ' $a_{(k+1)+1}$ ' by ' b_{k+1} '.]

[illegible]

$$[84] = \frac{1}{2} \sum_{i=0}^{\infty} i(i+1) \binom{n-1-i}{i} x^{n-1-i} y^{i+1}. \quad (9)$$

$$Q = \{i_1, \dots, i_{\ell} \in \mathbb{N}_n\}.$$

2.

•

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1. $\mathcal{A} \in \mathcal{A}(\mathcal{H})$ is a \mathcal{K} -operator if and only if $\mathcal{A}^* \in \mathcal{A}(\mathcal{H})$ is a \mathcal{K} -operator.

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$$18. (a) 2 \quad (b) 2 \quad (c) 0 \quad (d) 1 [81 \cdot 9^j = 720 + 9^j \iff 80 \cdot 9^j = 720]$$

$$(e) 3 [4^k - 4^{k-1} = 4 \cdot 4^{k-1} - 4^{k-1} = (4-1)4^{k-1} = 48]$$

$$\star(f) 2 [3^{2k+2}(3^{k-2} - 1) - 3^6(3^{k-2} - 1) = 0]$$

$$\iff (3^{2k+2} - 3^6)(3^{k-2} - 1) = 0$$

$$\iff k = 2 \quad \text{or} \quad k = 2$$

$$\star(g) -5, -3 [(2^{k+5} - 1)(2^{k+3} - 1) = 0]$$

$$19. (a) (2^{2^i} + 1)(2^{2^i} - 1) = (2^{2^i})^2 - 1^2 = 2^{2^i \cdot 2} - 1 = 2^{2^{i+1}} - 1$$

$$(b) (i) \prod_{i=0}^{-1} (2^{2^i} + 1) = 1; \quad 2^{2^{-1+1}} - 1 = 2^{2^0} - 1 = 2^1 - 1 = 1$$

$$\begin{aligned} (ii) \prod_{i=0}^{k+1} (2^{2^i} + 1) &= \prod_{i=0}^k (2^{2^i} + 1)(2^{2^{k+1}} + 1) \\ &= (2^{2^{k+1}} - 1)(2^{2^{k+1}} + 1) \\ &= 2^{2^{k+2}} - 1 \end{aligned}$$

$$20. (a) a_1 [\text{or: } a_1^1]; \quad a_2^{a_1}; \quad a_3^{a_2^{a_1}}$$

$$(b) 2^{2^{2^2}} = 2^{2^4} = 2^{16} = 65536$$

$$(c) (i) \prod_{p=1}^{0+1} a_p = \prod_{p=1}^1 a_p = a_1 = 1; \quad \prod_{p=1}^0 a_{p+1} = 1$$

$$21. \forall_n \prod_{p=1}^n \left(1 - \frac{1}{p+1}\right) = \frac{1}{n+1} \quad \left[1 - \frac{1}{p+1} = \frac{p}{p+1}, \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \dots \cdot \frac{n}{n+1} = \frac{1}{n+1}\right]$$

$$(i) \prod_{p=1}^1 \left(1 - \frac{1}{p+1}\right) = 1 - \frac{1}{1+1} = \frac{1}{2} = \frac{1}{1+1}$$

$$\begin{aligned} (ii) \prod_{p=1}^{q+1} \left(1 - \frac{1}{p+1}\right) &= \prod_{p=1}^q \left(1 - \frac{1}{p+1}\right) \left(1 - \frac{1}{q+2}\right) \\ &= \frac{1}{q+1} \left(1 - \frac{1}{q+2}\right) \\ &= \frac{1}{q+1} \left(\frac{q+1}{q+2}\right) = \frac{1}{q+2} \end{aligned}$$

$$22. (i) \sum_{p=1}^1 a_p = a_1. \text{ Since } \forall_p a_p \in I^+, a_1 \in I^+.$$

$$(ii) \sum_{p=1}^{q+1} a_p = \sum_{p=1}^q a_p + a_{q+1}. \text{ By the inductive hypothesis } \sum_{p=1}^q a_p \in I^+.$$

And, since $\forall_p a_p \in I^+$, $a_{q+1} \in I^+$. So, since I^+ is closed with

respect to addition, $\sum_{p=1}^{q+1} a_p \in I^+.$

[Variations on Exercise 22 include:

$$\forall_p a_p \in I^+ \Rightarrow \forall_n \prod_{p=1}^n a_p \in I^+$$

$$\forall_p a_p \in I \Rightarrow \forall_n \sum_{p=1}^n a_p \in I$$

$$\forall_p a_p \in I \Rightarrow \forall_n \prod_{p=1}^n a_p \in I \quad]$$

★23. (i) Since $a = a^1$ and $b = b^1$, it follows that if $a > b$ then $a^1 > b^1$.

(ii) Suppose, for $b > 0$, that if $a > b$ then $a^q > b^q$ [inductive hypothesis]. Suppose that $a > b$. It follows, for $b > 0$, that $a > 0$, and that $a^q > b^q$. Since $a > 0$, it follows, by Theorem 152a, that $a^q > 0$ and, since $a > b$, that $a^q a > a^q b$ --that is, that $a^{q+1} > a^q b$. Also, for $b > 0$, since $a^q > b^q$, it follows that $a^q b > b^q b = b^{q+1}$. Hence, by transitivity, $a^{q+1} > b^{q+1}$. Consequently, if $a > b$ then $a^{q+1} > b^{q+1}$. Etc.

$$24. (-1)^{-k} = 1/(-1)^k = 1^k/(-1)^k = (1/-1)^k = (-1)^k;$$

$$\text{or: } (-1)^{-k} = (-1)^{-k} \cdot 1 = (-1)^{-k} (-1)^{2k} = (-1)^{-k+2k} = (-1)^k$$

25. (a) If a and b are two positive numbers then the geometric mean, \sqrt{ab} , of a and b is the geometric mean, $\sqrt{\frac{a+b}{2} \cdot \frac{2ab}{a+b}}$, of their arithmetic mean and their harmonic mean. Since $\frac{a+b}{2} = \frac{2ab}{a+b}$ only if $(a+b)^2 = 4ab$ --that is, only if $(a-b)^2 = 0$, and since $a \neq b$, it follows that $\frac{a+b}{2} \neq \frac{2ab}{a+b}$. Since the geometric mean of two numbers is between the two numbers, it follows that the geometric mean of $\frac{a+b}{2}$ and $\frac{2ab}{a+b}$ is between $\frac{a+b}{2}$ and $\frac{2ab}{a+b}$. Hence, the geometric mean of a and b is between their arithmetic mean and their harmonic mean.

THEORY OF THE EARTH

CHAPTER I. OF THE ORIGIN OF THE EARTH.

The first question which presents itself to the mind is, what was the origin of the earth? and the answer to this question is, that it was created by the power of God, out of nothing, at the beginning of time. The earth was created in a state of confusion, and was afterwards divided into four parts, or continents, by the power of God. The first of these continents was Asia, which was created out of the earth, and was the first of the four continents to be inhabited by man. The second continent was Europe, which was created out of the sea, and was the second of the four continents to be inhabited by man. The third continent was Africa, which was created out of the earth, and was the third of the four continents to be inhabited by man. The fourth continent was America, which was created out of the sea, and was the fourth of the four continents to be inhabited by man.

The earth was created in a state of confusion, and was afterwards divided into four parts, or continents, by the power of God. The first of these continents was Asia, which was created out of the earth, and was the first of the four continents to be inhabited by man.

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- (b) Since the arithmetic mean of two positive numbers is greater than their geometric mean, it follows from part (a) that the harmonic mean of two positive numbers is less than their geometric mean and, so, is less than their arithmetic mean:

$$\forall x > 0 \quad \forall y > 0 \quad [x \neq y \Rightarrow \frac{2xy}{x+y} < \sqrt{xy} < \frac{x+y}{2}]$$

$$(c) \quad \frac{\frac{a+b}{2} + \sqrt{ab}}{2} = \frac{(\sqrt{a})^2 + 2\sqrt{ab} + (\sqrt{b})^2}{4} = \left(\frac{\sqrt{a} + \sqrt{b}}{2} \right)^2$$

$$(d) \quad \frac{2\sqrt{ab} \frac{2ab}{a+b}}{\sqrt{ab} + \frac{2ab}{a+b}} = \frac{4\sqrt{ab} (ab)}{\sqrt{ab} (a+b) + 2ab} = \frac{4(ab)}{a+b+2\sqrt{ab}} = \left(\frac{2\sqrt{a} \sqrt{b}}{\sqrt{a} + \sqrt{b}} \right)^2$$

- ★26. The least prime divisor of a composite number is at most equal to the square root of the number [Theorem 181]. Hence, the least prime divisor of a composite number which is less than 2500 is a prime number less than 50. There are, by count, 15 such prime numbers. Since $16 > 15$, it follows that, of any 16 composite numbers less than 2500, at least 2 will have the same least prime factor [see the pigeon-hole principle, page 8-84].

x_1, x_2, \dots, x_n are the roots of the equation $x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$.
 Then x_1, x_2, \dots, x_n are the roots of the equation $x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$.
 If x_1, x_2, \dots, x_n are the roots of the equation $x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$,
 then x_1, x_2, \dots, x_n are the roots of the equation $x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$.

x_1, x_2, \dots, x_n are the roots of the equation $x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$.

Then x_1, x_2, \dots, x_n are the roots of the equation $x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$.

x_1, x_2, \dots, x_n are the roots of the equation $x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$.

Then x_1, x_2, \dots, x_n are the roots of the equation $x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$.

x_1, x_2, \dots, x_n are the roots of the equation $x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$.

32. There is only one way. Suppose that the amounts paid are x_1, x_2, x_3, x_4, x_5 , and x_6 . Then, $x_1 = \frac{x_6 + x_2}{2}$, $x_2 = \frac{x_1 + x_3}{2}$, ..., and $x_6 = \frac{x_5 + x_1}{2}$. Suppose that $x_6 \leq x_2$. Since $x_1 = \frac{x_6 + x_2}{2}$, it follows that $x_1 \leq x_2 \leq x_3$. Continuing, we see that if $x_6 \leq x_2$ then

$$x_6 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq x_6.$$

Similarly, if $x_6 \geq x_2$ then

$$x_6 \geq x_1 \geq x_2 \geq x_3 \geq x_4 \geq x_5 \geq x_6.$$

In either case, by transitivity,

$$x_1 = x_2 = x_3 = x_4 = x_5 = x_6.$$

$$29. \quad (i) \quad \overbrace{\prod_{i=j}^{j-1} (a_i b_i)} = 1; \quad \overbrace{\prod_{i=j}^{j-1} a_i} \cdot \overbrace{\prod_{i=j}^{j-1} b_i} = 1 \cdot 1 = 1$$

$$\begin{aligned} (ii) \quad \overbrace{\prod_{i=j}^{k+1} (a_i b_i)} &= \overbrace{\prod_{i=j}^k (a_i b_i)} (a_{k+1} b_{k+1}) \\ &= \overbrace{\prod_{i=j}^k a_i} \cdot \overbrace{\prod_{i=j}^k b_i} \cdot (a_{k+1} b_{k+1}) \\ &= \overbrace{\prod_{i=j}^k a_i} \cdot a_{k+1} \cdot \overbrace{\prod_{i=j}^k b_i} \cdot b_{k+1} \\ &= \overbrace{\prod_{i=j}^{k+1} a_i} \cdot \overbrace{\prod_{i=j}^{k+1} b_i} \end{aligned}$$

$$30. \quad \frac{(2n)!}{2^n \cdot n!} \quad [\text{See Exercise 2 of Part B on page 8-128. There are}$$

$$(*) \quad C(2n, 2) \cdot C(2n-2, 2) \cdot C(2n-4, 2) \cdot \dots \cdot C(2, 2)$$

ways of choosing the couples in a specified order. Since the order of choosing the couples is irrelevant, the number of ways of pairing off $2n$ people is obtained by dividing $(*)$ by $n!$. Now, using Theorem 172, it is easy to see that the expression in $(*)$ can be simplified to $\frac{(2n)!}{2^n}$.]

$$31. \quad 15 \quad [0 + 1 + 2 + 3 + 4 + 5 = 15]$$

Let \mathcal{H} be a Hilbert space and \mathcal{H}^* its dual space. Let $\mathcal{H} \otimes \mathcal{H}^*$ be the tensor product of \mathcal{H} and \mathcal{H}^* . Let $\mathcal{H} \otimes \mathcal{H}^*$ be the tensor product of \mathcal{H} and \mathcal{H}^* .

$$(1 + p) = \frac{1 + p}{1 + p} = 1$$

$$(1 + p) = \frac{1 + p}{1 + p} = 1$$

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$$(1 + p) = \frac{1 + p}{1 + p} = 1$$

(ii) Suppose that $P_q = \frac{q^2 + q + 2}{2}$. Since, by the recursive definition (2), $P_{q+1} = P_q + (q + 1)$, it follows that

$$\begin{aligned} P_{q+1} &= \frac{q^2 + q + 2}{2} + (q + 1) \\ &= \frac{q^2 + q + 2 + 2(q + 1)}{2} \\ &= \frac{[q^2 + q + (q + 1)] + (q + 1) + 2}{2} \\ &= \frac{(q + 1)^2 + (q + 1) + 2}{2} \end{aligned}$$

Etc.

(b) (1) $S_1 = 2$; $S_2 = 4$; $S_3 = 8$

(2) $\begin{cases} S_1 = 2 \\ \forall_n S_{n+1} = S_n + P_n [= S_n + \frac{n^2 + n + 2}{2}] \end{cases}$

(3) Since $S_1 = 2$ and, $\forall_n (\Delta S)_n = \frac{n^2 + n + 2}{2}$, it follows, by Theorem 140, that, for each n ,

$$\begin{aligned} S_n &= 2 + \sum_{p=1}^{n-1} \frac{p^2 + p + 2}{2} \\ &= 2 + \sum_{p=1}^{n-1} \left[\frac{p(p+1)}{2} + 1 \right] \\ &= 2 + \frac{1}{2} \sum_{p=1}^{n-1} p(p+1) + \sum_{p=1}^{n-1} 1 \\ &= 2 + \frac{1}{2} \frac{(n-1)n(n+1)}{3} + n - 1 \\ &= (n+1) \left[1 + \frac{n(n-1)}{6} \right] \\ &= \frac{(n+1)(n^2 - n + 6)}{6} \end{aligned}$$

1990-1991 1992-1993 1994-1995 1996-1997 1998-1999 2000-2001 2002-2003 2004-2005 2006-2007 2008-2009 2010-2011 2012-2013 2014-2015 2016-2017 2018-2019 2020-2021 2022-2023 2024-2025 2026-2027 2028-2029 2030-2031 2032-2033 2034-2035 2036-2037 2038-2039 2040-2041 2042-2043 2044-2045 2046-2047 2048-2049 2050-2051 2052-2053 2054-2055 2056-2057 2058-2059 2060-2061 2062-2063 2064-2065 2066-2067 2068-2069 2070-2071 2072-2073 2074-2075 2076-2077 2078-2079 2080-2081 2082-2083 2084-2085 2086-2087 2088-2089 2090-2091 2092-2093 2094-2095 2096-2097 2098-2099 2100-2101 2102-2103 2104-2105 2106-2107 2108-2109 2110-2111 2112-2113 2114-2115 2116-2117 2118-2119 2120-2121 2122-2123 2124-2125 2126-2127 2128-2129 2130-2131 2132-2133 2134-2135 2136-2137 2138-2139 2140-2141 2142-2143 2144-2145 2146-2147 2148-2149 2150-2151 2152-2153 2154-2155 2156-2157 2158-2159 2160-2161 2162-2163 2164-2165 2166-2167 2168-2169 2170-2171 2172-2173 2174-2175 2176-2177 2178-2179 2180-2181 2182-2183 2184-2185 2186-2187 2188-2189 2190-2191 2192-2193 2194-2195 2196-2197 2198-2199 2200-2201 2202-2203 2204-2205 2206-2207 2208-2209 2210-2211 2212-2213 2214-2215 2216-2217 2218-2219 2220-2221 2222-2223 2224-2225 2226-2227 2228-2229 2230-2231 2232-2233 2234-2235 2236-2237 2238-2239 2240-2241 2242-2243 2244-2245 2246-2247 2248-2249 2250-2251 2252-2253 2254-2255 2256-2257 2258-2259 2260-2261 2262-2263 2264-2265 2266-2267 2268-2269 2270-2271 2272-2273 2274-2275 2276-2277 2278-2279 2280-2281 2282-2283 2284-2285 2286-2287 2288-2289 2290-2291 2292-2293 2294-2295 2296-2297 2298-2299 2300-2301 2302-2303 2304-2305 2306-2307 2308-2309 2310-2311 2312-2313 2314-2315 2316-2317 2318-2319 2320-2321 2322-2323 2324-2325 2326-2327 2328-2329 2330-2331 2332-2333 2334-2335 2336-2337 2338-2339 2340-2341 2342-2343 2344-2345 2346-2347 2348-2349 2350-2351 2352-2353 2354-2355 2356-2357 2358-2359 2360-2361 2362-2363 2364-2365 2366-2367 2368-2369 2370-2371 2372-2373 2374-2375 2376-2377 2378-2379 2380-2381 2382-2383 2384-2385 2386-2387 2388-2389 2390-2391 2392-2393 2394-2395 2396-2397 2398-2399 2400-2401 2402-2403 2404-2405 2406-2407 2408-2409 2410-2411 2412-2413 2414-2415 2416-2417 2418-2419 2420-2421 2422-2423 2424-2425 2426-2427 2428-2429 2430-2431 2432-2433 2434-2435 2436-2437 2438-2439 2440-2441 2442-2443 2444-2445 2446-2447 2448-2449 2450-2451 2452-2453 2454-2455 2456-2457 2458-2459 2460-2461 2462-2463 2464-2465 2466-2467 2468-2469 2470-2471 2472-2473 2474-2475 2476-2477 2478-2479 2480-2481 2482-2483 2484-2485 2486-2487 2488-2489 2490-2491 2492-2493 2494-2495 2496-2497 2498-2499 2500-2501 2502-2503 2504-2505 2506-2507 2508-2509 2510-2511 2512-2513 2514-2515 2516-2517 2518-2519 2520-2521 2522-2523 2524-2525 2526-2527 2528-2529 2530-2531 2532-2533 2534-2535 2536-2537 2538-2539 2540-2541 2542-2543 2544-2545 2546-2547 2548-2549 2550-2551 2552-2553 2554-2555 2556-2557 2558-2559 2560-2561 2562-2563 2564-2565 2566-2567 2568-2569 2570-2571 2572-2573 2574-2575 2576-2577 2578-2579 2580-2581 2582-2583 2584-2585 2586-2587 2588-2589 2590-2591 2592-2593 2594-2595 2596-2597 2598-2599 2600-2601 2602-2603 2604-2605 2606-2607 2608-2609 2610-2611 2612-2613 2614-2615 2616-2617 2618-2619 2620-2621 2622-2623 2624-2625 2626-2627 2628-2629 2630-2631 2632-2633 2634-2635 2636-2637 2638-2639 2640-2641 2642-2643 2644-2645 2646-2647 2648-2649 2650-2651 2652-2653 2654-2655 2656-2657 2658-2659 2660-2661 2662-2663 2664-2665 2666-2667 2668-2669 2670-2671 2672-2673 2674-2675 2676-2677 2678-2679 2680-2681 2682-2683 2684-2685 2686-2687 2688-2689 2690-2691 2692-2693 2694-2695 2696-2697 2698-2699 2700-2701 2702-2703 2704-2705 2706-2707 2708-2709 2710-2711 2712-2713 2714-2715 2716-2717 2718-2719 2720-2721 2722-2723 2724-2725 2726-2727 2728-2729 2730-2731 2732-2733 2734-2735 2736-2737 2738-2739 2740-2741 2742-2743 2744-2745 2746-2747 2748-2749 2750-2751 2752-2753 2754-2755 2756-2757 2758-2759 2760-2761 2762-2763 2764-2765 2766-2767 2768-2769 2770-2771 2772-2773 2774-2775 2776-2777 2778-2779 2780-2781 2782-2783 2784-2785 2786-2787 2788-2789 2790-2791 2792-2793 2794-2795 2796-2797 2798-2799 2800-2801 2802-2803 2804-2805 2806-2807 2808

Correction. On page 8-233, line 8b should begin:

31. What is the least number

- 27, (a) For each p , let a_p be the number of 1 foot squares which contain $p - 1$ blades of grass. We wish to show that, for some m , $a_m \geq 2$. For some positive integer, say n , $a_p = 0$ for $p > n$ --that is, no 1 foot square contains as many as n blades of grass.

For this n , $\sum_{p=1}^n a_p = 40000$, the total number of 1 foot square regions. It follows, by Theorem 144, that, for some $m \leq n$, $a_m \geq \frac{40000}{n}$. From this it follows [since each a_p is an integer] that, for some $m \leq n$, $a_m \geq 2$ if $\frac{40000}{n} > 1$ --that is, if $n < 40000$. So, if no 1 foot square contains as many as 39999 blades of grass, it is the case some two such squares contain the same number of blades of grass. Now, a few experiments will convince one that no 1 foot square region can contain 39999 blades of grass. In fact, another use of Theorem 144 shows that if some such region contains 39999 blades of grass then some 1 square inch region contains at least $39999/144$ blades. Since $39999/144 \doteq 278$, this is clearly impossible.

- (b) All that matters is that each of the regions into which the lawn is divided contains less than 39999 grass blades.

☆ 28 (a) (1) $P_1 = 2$; $P_2 = 4$; $P_3 = 7$

(2) $\begin{cases} P_1 = 2 \\ \forall_n P_{n+1} = P_n + (n + 1) \end{cases}$

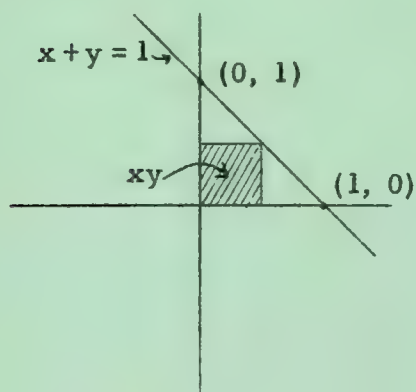
(3) $\forall_n P_n = \frac{n^2 + n + 2}{2}$ [Comparison with the recursive definition of T on page 7-61 of Unit 7 suggests that, for each n , $P_n = T_n + 1$. Now, use the explicit definition of T on page 7-63.]

(i) $P_1 = 2 = \frac{1^2 + 1 + 2}{2}$

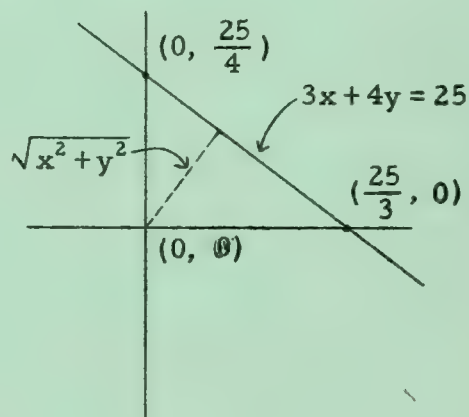
33. (a) $\frac{1}{4}$ [If $x + y = 1$ then $xy = x(1 - x) = \frac{1}{4} - \left(\frac{1}{2} - x\right)^2$. Hence, if $x + y = 1$ then $xy \leq \frac{1}{4}$ and, for $x = \frac{1}{2}$, $xy = \frac{1}{4}$.]

(b) 5 [If $3x + 4y = 25$ then $x^2 + y^2 = x^2 + \left(\frac{25 - 3x}{4}\right)^2 = \frac{25}{16}[(x - 3)^2 + 16]$. Hence, if $3x + 4y = 25$ then $\sqrt{x^2 + y^2} \geq 5$ and, for $x = 3$, $\sqrt{x^2 + y^2} = 5$.]

[The geometric interpretations of the extrema problems are interesting. In (a) we are trying to find the rectangle of greatest area-measure whose upper-right vertex belongs to the graph of ' $x + y = 1$ '. In (b) we are trying to find the shortest distance from $(0, 0)$ to the graph of ' $3x + 4y = 25$ '.



the maximum occurs
when the rectangle is
a square



the minimum occurs when
the radius vector is per-
pendicular to the line

$$34. \quad ab^2 + 2b \left[a = x^{-2} + y^{-2} = \frac{x^2 + y^2}{(xy)^2} = \frac{x^2 + y^2}{b^2}; (x + y)^2 = x^2 + y^2 + 2xy \right. \\ \left. = ab^2 + 2b \right]$$

35. $40 [= 5 \cdot 2 \cdot 4]; \quad 22 [= 3 \cdot 2 \cdot 3 + 2 \cdot 1 \cdot 2]; \quad 24 [= 3 \cdot 2 \cdot 4]$

36. (a) $9!$ (b) $9! \cdot 2$ (c) $9!(2^{10} - 3) [= 9! 2^{10} - (9! + 9! \cdot 2)]$

37. (a) $3600 [= 7! - 6! \cdot 2]$ (b) $4320 [= 6 \cdot 6!]$ (c) $720 [= 6!]$

38. (a) $64 [= C(10, 3) - C(8, 3)]$ (b) $8 [= C(8, 1)]$
 (c) $56 [= C(8, 3)]$ (d) $112 [= C(8, 3) + 2 \cdot C(8, 2)]$

39. (a) $725760 [= 2 \cdot 9!]$ (b) $2903040 [= 10! - 2 \cdot 9!]$

$\{1, 2, \dots, n\}$ $\{1, 2, \dots, n\}$ $\{1, 2, \dots, n\}$ $\{1, 2, \dots, n\}$

$\{1, 2, \dots, n\}$ $\{1, 2, \dots, n\}$ $\{1, 2, \dots, n\}$ $\{1, 2, \dots, n\}$

$\{1, 2, \dots, n\}$ $\{1, 2, \dots, n\}$ $\{1, 2, \dots, n\}$ $\{1, 2, \dots, n\}$

$\{1, 2, \dots, n\}$ $\{1, 2, \dots, n\}$ $\{1, 2, \dots, n\}$ $\{1, 2, \dots, n\}$

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n = 0 \quad (1)$$

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n = 0$$

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n = 0$$

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$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n = 0$$

$$\star 45. \quad a_n = 1 \cdot 10^{2n-1} + 1 \cdot 10^{2n-2} + \dots + 1 \cdot 10^n + 5 \cdot 10^{n-1} + 5 \cdot 10^{n-2} \\ + \dots + 5 \cdot 10^1 + 6$$

$$= \sum_{p=n}^{2n-1} 10^p + 5 \sum_{p=1}^{n-1} 10^p + 6$$

$$= \sum_{p=1}^n 10^{p+n-1} + 50 \sum_{p=1}^{n-1} 10^{p-1} + 6$$

$$= 10^n \sum_{p=1}^n 10^{p-1} + 50 \sum_{p=1}^{n-1} 10^{p-1} + 6$$

$$= 10^n \frac{10^n - 1}{9} + 50 \frac{10^{n-1} - 1}{9} + 6$$

$$= \frac{10^{2n} - 10^n + 5 \cdot 10^n - 50 + 54}{9}$$

$$= \frac{10^{2n} + 4 \cdot 10^n + 4}{9} = \left(\frac{10^n + 2}{3} \right)^2$$

It is easy to show that $3 \mid 10^n + 2$. One can use induction:

$$3 \mid 10^1 + 2; \quad 10^{n+1} + 2 = 10^n \cdot 10 + 2 = 10^n + 9 \cdot 10^n + 2 \\ = (10^n + 2) + 9 \cdot 10^n$$

or the binomial theorem:

$$10^n + 2 = (9 + 1)^n + 2 = 9k + 1^n + 2 = 9k + 3$$

Hence, a_n is the square of an integer.

Moreover, $(\sqrt{a})_n = \frac{10^n + 2}{3}$. So,

$$\sqrt{a}: 4, 34, 334, \dots$$

Correction. On page 8-225, Exercise

4(e) should be: $n^2 - n - 2 \sum_{p=1}^{10} p$

$$40. \quad 5 \left[\sum_{p=1}^{10} [5(a_p - 4) - 4] = 5 \sum_{p=1}^{10} a_p - \sum_{p=1}^{10} 24 = 245 - 240 \right]$$

$$41. \quad 485 - 198\sqrt{6}$$

42. Each of the terms contains an 'x'.

[The $(k+1)$ th term is ' $C(18, k)(x^5)^{18-k}(x^{-2})^k$ ', or, more simply, ' $C(18, k) x^{90-7k}$ '. There is no k such that $90 - 7k = 0$.]

$$43. \quad -12446720 x^8 y^9 \quad [= C(17, 9)x^{17-9}(-2y)^9 = -24310x^8 \cdot 2^9 y^9]$$

$$44. \quad (a) \quad y [4y^2(x - 3y) - 1]$$

$$(b) \quad (1 - x - 2y)(1 + x + 2y)$$

$$(c) \quad (x + z^{-1})(x^4 - x^3 z^{-1} + x^2 z^{-2} - x z^{-3} + z^{-4}) \quad (d) \quad (x + x^{-1})^2$$

$$(e) \quad (n - 11)(n + 10)$$

$$(f) \quad 2n(n + 1)$$

$$(g) \quad (7 - 2x)(3 + x)$$

$$(h) \quad -(x - 1)(x + 2)(x^2 - x + 2)$$

$$(i) \quad (x - y)(x^2 + xy + y^2 + x + y)$$

$$(j) \quad (a - b)(a + b)(a^2 + b^2)(a^4 + b^4)$$

$$(k) \quad 4(a - 2)(a^4 + 2a^3 + 4a^2 + 8a + 16)$$

$$(l) \quad \pi(R - r)(R + r)$$

$$(m) \quad x^2 y(y - x)(y + x)$$

$$(n) \quad x(x + 1)^2$$

★46. Guess: $a_n = n^3$

Proof: $a_n = n \sum_{p=1}^n (2p - 1) = n \cdot n^2 = n^3$

47. $4 [(2^k - 16)(2^k - 5) = 0]$

48. (a) 7

(b) -4

(c) 0.06

49. $10 \frac{10}{11}$ minutes after 2 o'clock

[Suppose that at x minutes after 2 o'clock but before 3 o'clock the hands are pointing in the same direction. In x minutes the tip of the minute hand has moved $x/60$ of a trip around the clock, or $\frac{x}{60} - \frac{2}{12}$ of its trip between 2 and 3. In the same time the hour hand has moved $\frac{x}{60}$ of its trip between 2 and 3. So, $\frac{x}{60} - \frac{2}{12} = \frac{x}{60} \cdot \frac{1}{12}$.]

★50. (a) Suppose that d_n is the shortest distance the point can move during the n th second. Then,

$$\begin{aligned} d_n &= \sqrt{(x_n - x_{n-1})^2 + (y_n - y_{n-1})^2} \\ &= \sqrt{9n^2 + 16n^2} \\ &= 5n \end{aligned}$$

So, $\sum_{p=1}^n d_n = 5 \sum_{p=1}^n p = \frac{5n(n+1)}{2}$.

$$(b) \ x_{2n} = \sum_{p=1}^{2n} (x_p - x_{p-1}) = \sum_{p=1}^{2n} 3p = 3 \frac{2n(2n+1)}{2} = 3n(2n+1)$$

$$y_{2n} = \sum_{p=1}^{2n} (y_p - y_{p-1}) = \sum_{p=1}^{2n} (-1)^{p-1} 4p$$

$$= \sum_{q=1}^n (-1)^{(2q-1)-1} 4(2q-1) + \sum_{q=1}^n (-1)^{2q-1} 4(2q)$$

$$= \sum_{p=1}^n [4(2q-1) - 4(2q)] = -4n$$

51. (a) 3, -3

(b) $-\sqrt{7}, \sqrt{7}$

(c) $\frac{9}{4}$

52. (a) $\frac{5}{11}$

(b) $\frac{33}{65}$

(c) $\frac{21}{10}$

53. $2\frac{26}{27}$ gallons $[= \left(\frac{8}{12}\right)^3 \cdot 10 \text{ gallons}]$

$$MA = \frac{1}{\cos 40^\circ}, \quad NA = \frac{1}{\cos 30^\circ}, \quad (MN)^2 = (BM)^2 + (BN)^2 \\ = (\tan 40^\circ)^2 + (\tan 30^\circ)^2$$

So,

$$\cos (\angle A) = \frac{\left(\frac{1}{\cos 40^\circ}\right)^2 + \left(\frac{1}{\cos 30^\circ}\right)^2 - [(\tan 40^\circ)^2 + (\tan 30^\circ)^2]}{2 \cdot \frac{1}{\cos 40^\circ} \cdot \frac{1}{\cos 30^\circ}}$$

Now, for each x between 0 and 90,

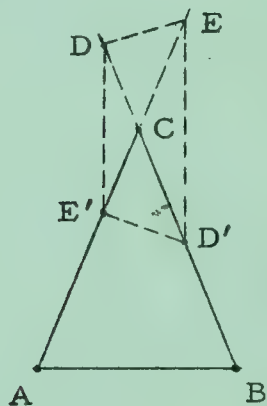
$$\tan x^\circ = \frac{\sin x^\circ}{\cos x^\circ} \text{ and } (\cos x^\circ)^2 = 1 - (\sin x^\circ)^2.$$

So,

$$\begin{aligned} \cos (\angle A) &= \frac{\frac{1 - (\sin 40^\circ)^2}{(\cos 40^\circ)^2} + \frac{1 - (\sin 30^\circ)^2}{(\cos 30^\circ)^2}}{2 \cdot \frac{1}{\cos 40^\circ} \cdot \frac{1}{\cos 30^\circ}} \\ &= \frac{\frac{1}{1} + \frac{1}{1}}{2 \cdot \frac{1}{\cos 40^\circ} \cdot \frac{1}{\cos 30^\circ}} = \cos 40^\circ \cdot \cos 30^\circ \\ &\doteq 0.766 \cdot 0.866 = 0.663356. \end{aligned}$$

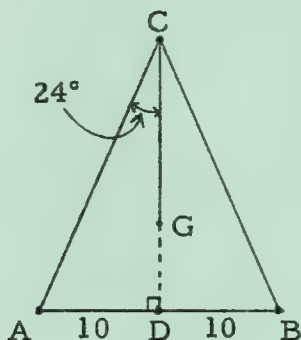
60. $1 \quad [a - b = b - a \iff a = b; \frac{1}{a} = 7 \Rightarrow a \neq 0]$

61. 45 or 135



Since $m(\angle ABC) + m(\angle CAB) + m(\angle ACB) = 180$, it follows from the hypothesis that $\angle ACB$ is an angle of 45° . There are four cases to consider: $C \in \overline{AE}$ and $C \in \overline{BD}$, $C \in \overline{AE}$ and $D \in \overline{BC}$, $E \in \overline{AC}$ and $C \in \overline{BD}$, and: $E \in \overline{AC}$ and $D \in \overline{BC}$. In the first and fourth cases, $m(\angle CDE) + m(\angle CED) = 180 - 45 = 135$. In the second and third cases, $m(\angle CDE) + m(\angle CED) = 45$.

56. 15 inches



$$CD = 10 \cdot \tan 66^\circ \doteq 22.46$$

$$CG = \frac{2}{3} \cdot CD \doteq 14.97$$

57. (a) 55 [$x + 10 + x - 30 = 90$]

(b) 18 [$4x + x = 90$]

(c) 13 [$3x + 6 = 45$]

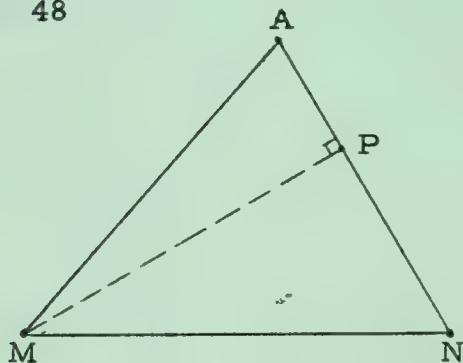
(d) 6 [$5x = 30$]

(e) 20 [$3x = 60$]

58. 35

[The angle in question is $\angle GAF$ [see figure for Exercise 59]. Since $\angle AFG$ is a right angle, $\tan (\angle GAF) = GF/AF = 1/\sqrt{2}$. So, $\tan (\angle GAF) \doteq 0.707$. Hence, $^{\circ}m(\angle GAF) \doteq 35$.]

☆59. 48



$$[(MN)^2 - (NA - PA)^2] + (PA)^2 = (MA)^2$$

$$(MN)^2 - (NA)^2 + 2 \cdot NA \cdot PA = (MA)^2$$

$$PA = \frac{(MA)^2 + (NA)^2 - (MN)^2}{2 \cdot NA}$$

$$\cos (\angle A) = PA/MA$$

$$= \frac{(MA)^2 + (NA)^2 - (MN)^2}{2 \cdot MA \cdot NA}$$

Suppose that $AB = 1$. Then,

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1. *Chlorophyll a* (Chl *a*)

1990

1

[illegible]

1. *Journal of the American Medical Association*, 1990; 263: 1039-1041.

$$f_{\text{max}} = 100 \text{ Hz} \quad \text{and} \quad f_{\text{min}} = 10 \text{ Hz} \quad \text{and} \quad f_{\text{max}} + f_{\text{min}} = 110 \text{ Hz} \quad (1)$$

$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$ $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix}$ $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$

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• 1997 •

2011

2. 3. 16

1. *Chlorophyll a* and *Chlorophyll b* contents were determined by spectrophotometry using the method of Lichtenthal and Whistler (1987).

1947

2015-11-11

[illegible]

161. 23. 5.

9. 1. 1. 1.

$$1 \leq i \leq n, \quad 1 \leq j \leq n, \quad 1 \leq k \leq n,$$

1990

1000 1000 1000 1000 1000

• • • • •

1. *Chlorophyll a* and *Chlorophyll b* were determined by the method of Arar and Collins (1971) using a Shimadzu 1601 UV-Visible Spectrophotometer.

